5 Stars and Stellar Structure

5.1 Phenomenology of stars

Almost all the light we see in the Universe comes from stars, either directly, or else indirectly from the fact that stars heat the surrounding gas and dust. Almost the only exceptions to this rule are

- non-thermal radiation (synchrotron radiation, for example) from high energy particles spiraling around in magnetic fields
- emission from quasars, thought to be from the gravitational potential energy released as matter spirals, via an accretion disk, into a black hole.

We have already seen that stars can be classified, purely empirically, on the basis of their spectra: O B A F G K M R N S. This is pretty much a classification by surface temperature, but not entirely. More sophisticated application of physics to a spectrum allows one to determine separately

- the temperature (which lines, and which ionization states are present)
- the surface gravity (at a given temperature, a line will be more pressure-broadened if the surface gravity is larger)
- chemistry or elemental abundances (relative strength of lines from different elements)

5.1.1 Elemental abundances, populations I and II

The last classification above, elemental abundances, reveals things not only about stars in isolation, but also things about the history of the Universe, i.e.

cosmology. To refresh your memory about the elements, here is the Periodic Table.

The numbers shown on the periodic table are the atomic numbers or numbers of protons, Z, in the nucleus. This of course determines the number of electrons and hence the chemistry. For *nuclear* processes, however, we also care about the total number of nucleons (protons plus neutrons), A, in the nucleus. For all the elements from He (not H!) through S (sulfur), the relationship is

$$A \approx 2Z$$

within one AMU. That is, there are close to equal numbers of protons and neutrons. Heavier elements get slightly richer in neutrons so A becomes somewhat larger than $2Z$ (but not much). Iron, for example has $Z = 26, A \approx 56$.

Studying their spectra, it is found that there are two distinct classes of stars, called Population I and Population II (usually read as "Pop One" and "Pop Two"). Pop I stars are young stars, like the Sun, created by ongoing star-formation processes within our Galaxy (or in other galaxies). Pop II stars are old stars and are thought to fossilize the *initial epoch* of star formation after the Big Bang.

When this classification was first devised, the connection with age was not immediately obvious, so the numbering scheme ("I" and "II") is perhaps illogical. A good way to remember it is to recall that a II-year-old child is *older* than a I-year-old child.

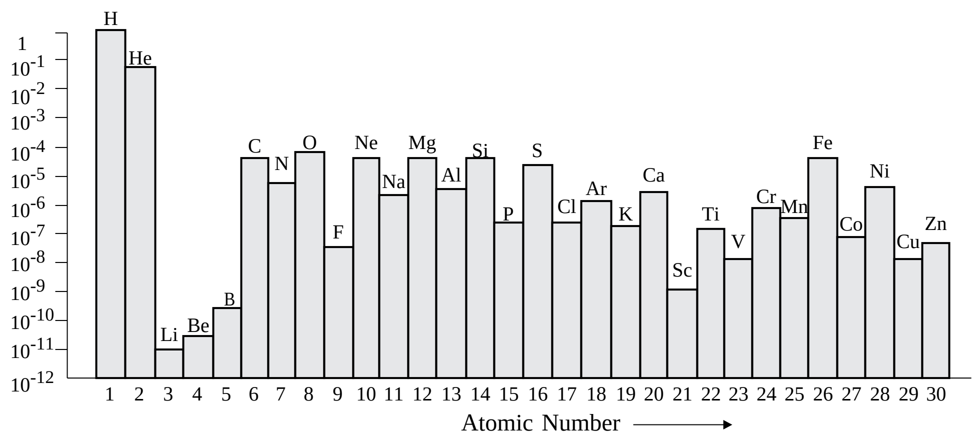
The observational distinction between Pop I and Pop II is that Pop I stars (e.g., the Sun) are relatively rich in "heavier" elements. For astronomers, "heavier" means anything beyond H and He in the periodic table, that is, Li, Be, B, C, N, O, etc. Chemists would consider these to be light elements! Another astronomer's habit of which you should be aware is that the "heavy" (though not really heavy!) elements are often referred to loosely as "metals", even though they're not. So, to astronomers, the whole periodic table is reduced to H, He, and "metals".

We know quite a lot about the elemental abundance of Pop I stars, because we have one right in our neighborhood (the Sun), and also because, except for H and He which escaped into space, the Earth itself has in most respects the exact elemental abundances of a typical Pop I star. Here is a table, and also graph (note logarithmic scale!) of these "Solar System abundances".

Pop I Abundances

	Atomic Number	Mass (main isotope)	Relative Number	Mass Fraction
H	1	1	1	0.77
He	2	4	7×10^{-2}	0.21
C	6	12	4×10^{-4}	4×10^{-3}
N	7	14	9×10^{-5}	1×10^{-3}
O	8	16	7×10^{-4}	9×10^{-3}
Ne	10	20	1×10^{-4}	1×10^{-3}
Mg	12	24	4×10^{-5}	8×10^{-4}
Si	14	28	4×10^{-5}	8×10^{-4}
Fe	26	56	3×10^{-5}	1×10^{-3}

“metals” total 0.02.



Hydrogen and Helium are primordial to the big bang (in fact, the Helium is produced during the first few minutes of the big bang). All the elements from Carbon on are produced *solely* in stars. The combined mass fraction of “metals” (Carbon and heavier) is usually denoted Z, so Pop I stars have typically $Z \approx 0.02$. (Don’t confuse this use of the letter Z with its other use, denoting atomic number. They are unrelated.)

Note the very low abundances of Li, Be, B. These are hardly produced in the big bang at all, and they are in fact *destroyed* by stars – they get “cooked” into heavier elements. Also note that the even elements tend to be more abundant (by a factor of order 10) than their odd neighbors. This is because they get "built" efficiently out of α-particles (Helium nucleii) that contain 2 protons + 2 neutrons.

So, Pop I stars are made of material that has *already been processed* through an earlier generation of stars, either Pop II stars or else older Pop I stars (whose creation and destruction is a continuous process). In fact, many astrophysicists study what amounts to the "ecology" of stars, that is, processes by which they form, evolve, are disrupted, and return their now-processed material to the interstellar medium for formation in a subsequent generation of stars.

Since Pop II stars were (as far as we know) the original generation of stars, you would expect them to have much lower metal abundances. Indeed that is true. Typically they might have $Z \approx 0.002$, about 1/10 of the Solar abundances. Some have even smaller Z values.

In our Galaxy, Pop I stars are located in the disk, while Pop II stars are in the so-called bulge and halo. Thus, we know that these parts of the Galaxy, being populated with primordial stars, are older.

5.1.2 Nuclear reactions

Stars are powered by nuclear reactions that transmute (we sometimes loosely say "burn") lighter elements into heavier ones. Main-sequence stars are all powered by the simplest possible transmutation, namely of four Hydrogen nucleii (protons) into one Helium nucleus.

The reason that burning H to He produces energy is of course the fact that the He nucleus weighs slightly less than the 4 H's. In atomic mass units (AMUs):

$$4 \times M_{\rm H} = 4 \times 1.008 \rightarrow 1 \times M_{\rm He} = 4.0027 \ .$$

That is

$$\frac{\Delta M}{4M_{\rm H}} = \frac{4.032 - 4.0027}{4.032} = 0.007 \ .$$

So, 0.7% of the mass of each proton is converted to energy ($E = mc^2$).

In practice, because of the various rules governing nuclear reactions and their probabilities, the reaction is *not* simply

$$^1{\rm H} + {}^1{\rm H} + {}^1{\rm H} + {}^1{\rm H} \rightarrow {}^4{\rm He} \qquad \text{[not!]}$$

(The leading superscripts are used to indicate mass number.) For one thing, the above does not conserve charge! Instead, the reaction proceeds by a series of 2-body interactions. In lower mass stars ($< 1.5 M_\odot$) the so-called $p-p$ *cycle* dominates:

$$\begin{aligned}
{}^1{\rm H} + {}^1{\rm H} &\longrightarrow {}^2D + e^+ + \nu \\
{}^2{\rm D} + {}^1{\rm H} &\longrightarrow {}^3{\rm He} + \gamma \\
{}^3{\rm He} + {}^3{\rm He} &\longrightarrow {}^4{\rm He} + {}^1{\rm H} + {}^1{\rm H}
\end{aligned}$$

Higher mass stars ($> 1.5 M_\odot$) "burn" via the *CNO cycle*:

$$\begin{aligned}
{}^{12}{\rm C} + {}^1\mathbf{H} &\longrightarrow {}^{13}{\rm N} + \gamma \\
{}^{13}{\rm N} &\xrightarrow{\text{decays}} {}^{13}{\rm C} + \boldsymbol{e^+} + \boldsymbol{\nu} \\
{}^{13}{\rm C} + {}^1\mathbf{H} &\longrightarrow {}^{14}{\rm N} + \gamma \\
{}^{14}{\rm N} + {}^1\mathbf{H} &\longrightarrow {}^{15}{\rm O} + \gamma \\
{}^{15}{\rm O} &\xrightarrow{\text{decays}} {}^{15}{\rm N} + \boldsymbol{e^+} + \boldsymbol{\nu} \\
{}^{15}{\rm N} + {}^1\mathbf{H} &\longrightarrow {}^{12}{\rm C} + {}^4\mathbf{He} \ .
\end{aligned}$$

Notice that the ^{12}C is a *catalyst* in this case: it comes into the cycle at the top and is regenerated at the bottom. The net reaction involves only the particles shown above in boldface, namely

$$4 \times {}^1\mathrm{H} \longrightarrow {}^4\mathrm{He} + 2e^+ + 2\nu \ .$$

(The γ's are just energy coming off.)

The key fact about nuclear reactions is that they are *extremely* temperature sensitive. That is, above a certain threshold, a small increase in temperature makes a huge increase in reaction rate. This fact makes most stars thermally very constant: across a wide range of stellar masses, the central temperatures of stars (where the nuclear burning takes place) are between 1×10^7 K and 2×10^7 K. That is, this small temperature range translates into the entire range of luminosity needed to "hold up" against gravity both massive stars (high luminosity) and light stars (low luminosity). A good empirical approximation for the central temperature of main-sequence stars is

$$T_c \approx 1.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^{1/3} \mathrm{K} \ .$$

5.2 Stellar structure

5.2.1 Order-of-magnitude stellar structure

The virial theorem, which we previously derived for a collection of gravitating bodies, applies equally well to any combination of inverse-square-law forces between bodies. Since the nucleii and electrons in a star interact almost exclusively by Coulomb electromagnetic forces (and gravity), the virial theorem applies to them. Thus, in a star, the total kinetic energy of particles is exactly half the gravitational potential energy. (There is no electromagnetic potential energy to speak of, because of bulk charge neutrality).

We can estimate these two terms as follows:

$$\begin{aligned} \text{P.E.} &= \iint \frac{G dm_1 dm_2}{r_{12}} \sim \frac{GM^2}{R} \\ \text{K.E.} &= \frac{3}{2} N_{\text{particles}} kT \sim \left(\frac{M}{m_p}\right) kT \ . \end{aligned}$$

Here M is the star's mass, R its characteristic size (radius, say), T its characteristic (central T_c, say) temperature. Twiddles mean equality in order of magnitude, i.e. ignoring numerical constants like 2 or π.

Equating (at twiddle accuracy) P.E. and K.E., and using the empirical law for T_c given previously (which was motivated by the temperature sensitivity of nuclear reactions), we get a *mass-radius relation* for stars,

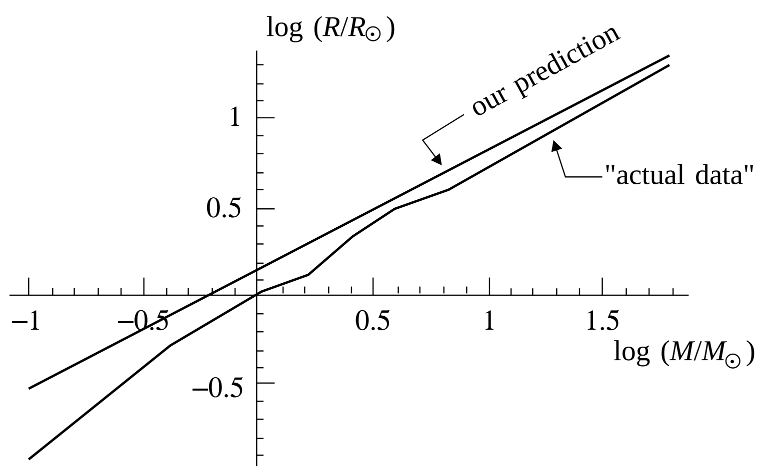
$$\begin{aligned} R \sim \frac{GMm_p}{kT} &\sim \frac{GM_\odot m_p}{k(15 \times 10^6\,\text{K})} \left(\frac{M}{M_\odot}\right)^{2/3} \\ &= \frac{(6.67 \times 10^{-8})}{(6 \times 10^{23})(1.38 \times 10^{-16})(15 \times 10^6)} \left(\frac{M}{M_\odot}\right) \text{ cm} \\ &= 1.0 \times 10^{11} \left(\frac{M}{M_\odot}\right)^{2/3} \text{ cm} \ . \end{aligned}$$

The actual solar radius is 7×10^{10}, so our twiddle calculation is actually pretty good (the neglected numerical factors cancel to near-unity)! How do we do for other stars on the main sequence? Here is some actual data:

Physical Properties of Main-Sequence Stars

$\log(M/M_\odot)$	Spectral class	$\log(L/L_\odot)$	$M_{\rm bol}$	M_V	$\log(R/R_\odot)$
−1.0	M6	−2.9	12.1	15.5	−0.9
−0.8	M5	−2.5	10.9	13.9	−0.7
−0.6	M4	−2.0	9.7	12.2	−0.5
−0.4	M2	−1.5	8.4	10.2	−0.3
−0.2	K5	−0.8	6.6	7.5	−0.14
0.0	G2	0.0	4.7	4.8	0.00
0.2	F0	0.8	2.7	2.7	0.10
0.4	A2	1.6	0.7	1.1	0.32
0.6	B8	2.3	−1.1	−0.2	0.49
0.8	B5	3.0	−2.9	−1.1	0.58
1.0	B3	3.7	−4.6	−2.2	0.72
1.2	B0	4.4	−6.3	−3.4	0.86
1.4	O8	4.9	−7.6	−4.6	1.00
1.6	O5	5.4	−8.9	−5.6	1.15
1.8	O4	6.0	−10.2	−6.3	1.3

Here is the comparison with our twiddle model:



Mass-Radius Relation for Stars

What about the *luminosity* of stars? Can we predict that with twiddle calculations? Yes, but we need to know something about the *opacity* of stellar material, that is, how much resistance it gives to the outward diffusion of photons. I will write down the calculation, even though it is beyond the scope of this course. You can study it for extra credit (or intellectual curiosity).

$$\frac{L}{4\pi r^2} = \text{Flux} = \left[\begin{array}{c}\text{Diffusion}\\ \text{Coefficient}\end{array}\right] \times \left[\begin{array}{c}\text{Energy Density}\\ \text{Gradient}\end{array}\right] = \left(\frac{c}{3\kappa\rho}\right)\frac{d(aT^4)}{dr} \ .$$

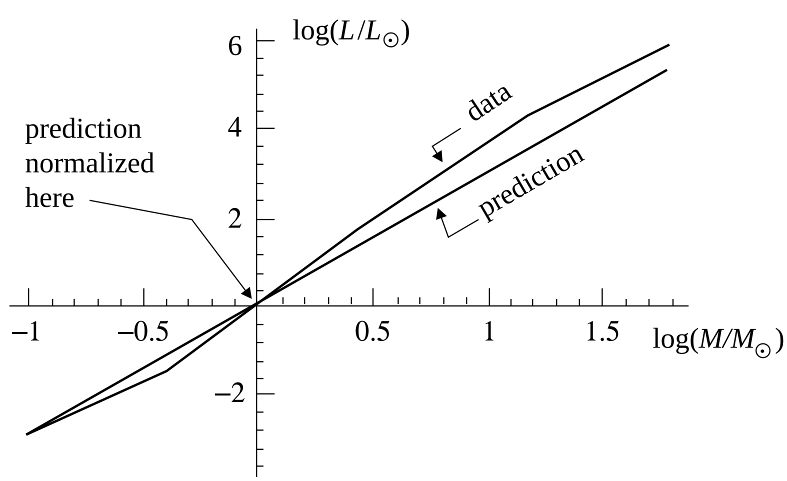
Here κ is the opacity, which comes from atomic physics. Let us assume that this is a constant (as it very nearly is for highly ionized matter). Then, neglecting constants, we have

$$\begin{aligned} &\rho \sim M/R^3 \\ &T \sim M/R \qquad \text{(virial theorem)} \ . \end{aligned}$$

So

$$L \propto \frac{R^2}{\rho}\frac{T^4}{R} \sim \frac{R^5}{M}\frac{(M/R)^4}{R} \sim M^3 \ .$$

Notice that the R's cancel, so we never had to use the mass-radius relation, or the empirical formula for central temperature, but only the virial theorem. How good is this? Since we do not at this stage know a numerical value for κ, we cannot check the constant, but only the scaling from the solar value:



Mass-Luminosity Relation for Stars

Pretty good! So it looks like stars really do obey the laws of physics. This motivates us to do a more careful job of writing down their governing "equations of stellar structure," which we will do next.

5.2.2 Quantities describing the stellar interior

What is a star? A self-gravitating gaseous system. Why call it a gas? It is so hot that all the original material inside has long since ionized to nuclei and electrons — *a plasma.* But for our purposes, it will be sufficient to describe this as a fully ionized *gas* of electrons plus ions (plus photons — in such a gas, "radiation pressure" may be important), which is entirely neutral — there are no significant internal dynamics of the plasma.

The simplest description of stellar structure is that the stars are *spherical* and *static* (no rotation, magnetic field, no pulsation, oscillation,...), and we will only deal with such objects here. In other words, stars involve the interplay of gravitation, gas dynamics, and radiation.

The basic idea is to write down quantities that describe the stellar interior as a function of radius r, then to write down relations between them, either algebraic, or else differential equations, until we get a *closed set* of equations (as many equations as unknowns). Then we can think about solving them.

Here is a list of such quantities:

$\rho(r)$ density: At the center of the star this will have some value ρ_c and decrease to zero at the surface of the star.

$M(r)$ mass: This is the mass *interior* to a radius r. It comes into the calculation of the gravitational force as a function of r. It is zero at $r = 0$, and equal to the star's total mass M_0 at the stellar surface.

$P(r)$ pressure: The pressure at any r is of course the total weight (per square centimeter) of all the *overyling* mass.

$T(r)$ temperature: Gas and radiation are in local thermodynamic equilibrium.

$L(r)$ luminosity: This is the net outward total energy flow at each radius. It grows from zero to the star's total luminosity as we pass from $r = 0$ through the star's energy generating region, and thereafter (as we move outward) is constant at the star's total luminosity.

There are also various quantities describing the microscopic local properties of the gas:

$\mu(r)$ mean particle mass (also called mean molecular weight): This comes into the perfect gas law $P = (\rho/\mu)kT$. It is the mean counting both electrons and nucleons as particles (since both contribute to pressure).

$\kappa(r)$ opacity or "mass absorption coefficient": This has units of cm^2/g and is the total cross sectional area for absorbing or scattering photons per gram of material. It controls the rate at which photons diffuse outward to transport the luminosity.

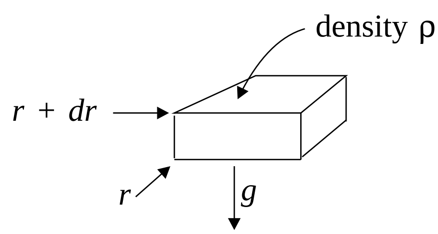
$\epsilon(r)$ The nuclear generation rate, in ergs per cm^3 per sec.

Now, if you are lazy, here is the good news: In *this course* we are going to use various tricks or approximations that close the set of equations with only the first three variables above: ρ, M, P. This "closure" is not exactly correct for all cases, but it will allow us to learn some interesting things about some real stars, and will let us defer the whole subject of "radiative transfer" (involving T, L, μ, K, and E) to later courses, Astronomy 145 and 150.

5.2.3 Equations of stellar structure

Equation of Hydrostatic Equilibrium

Consider an element of gas in equilibrium in the star



The pressure $P(r)$ is larger than the pressure $P(r+dr)$ by just the weight per unit area of the material between r and $r+dr$ in the local gravitational acceleration g. If the element is of area dA, we have

$$\begin{aligned}
\begin{pmatrix} \text{mass of} \\ \text{element} \end{pmatrix} &= \rho\, dA\, dr \\
\begin{pmatrix} \text{weight of} \\ \text{element} \end{pmatrix} &= g\rho\, dA\, dr = \left[\frac{GM(r)}{r^2}\right] \rho(r)\, dA\, dr \\
\begin{pmatrix} \text{weight of element} \\ \text{per unit area} \end{pmatrix} &= \frac{G\rho(r)M(r)}{r^2} dr\ .
\end{aligned}$$

So,

$$\frac{dP}{dr} = \frac{P(r+dr)-P(r)}{dr} = -\frac{G\rho(r)M(r)}{r^2}\ .$$

The minus sign is because pressure increases as r gets smaller (downward direction).

Mass Equation

Mass (interior to radius r) is just the integral of the density in spherical coordinates:

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 dr' \ .$$

We usually prefer to write this as a differential equation. Taking the derivative with respect to the upper limit of integration gives trivially

$$\frac{dM}{dr} = 4\pi r^2 \rho \ .$$

Equation of State

If we could just find an *algebraic* relation between pressure and density

$$P = P(\rho)$$

we would be done: 3 equations for the 3 unknowns P, ρ, M as a function of the independent variable r. In real life, however, pressure depends not only on density but also on *temperature* and *composition*. For a mixture of a perfect gas and radiation, we have

$$P = P_{\text{gas}} + P_{\text{radiation}} = \left(\frac{\rho}{\mu}\right) kT + \frac{1}{3} a T^4$$

(remember our calculation of radiation pressure in 3.3.4, and of radiation energy density in 3.5.4?). Luckily, it is often true that $P_{\text{radiation}} \ll P_{\text{gas}}$, so that the second term can be neglected, *and* that we can either (i) derive from other physics, or (ii) make a good guess, about how T in the first term varies with ρ. Further, in many cases of interest, μ is constant. Then, we will have arrived at a so-called *barytropic* equation of state $P = P(\rho)$. Let us now make that assumption. Later, we will catalog the actual cases for which it occurs.

5.2.4 Polytropes: The Lane-Emden equation

A *polytropic* equation of state is a special case of a barytropic equation of state where the relation between P and ρ is a pure power law

$$P = K\rho^{1+\frac{1}{n}} .$$

Here n (which need not be an integer) is the so-called polytropic index. (The weird notation "$1 + \frac{1}{n}$" can be thought of as arising from the perfect gas law

$$P \propto \rho T$$

along with an assumed power law relating T and ρ,

$$T \propto \rho^{1/n} \qquad \text{or} \qquad \rho \propto T^n .$$

But that is just notational history.)

It turns out that main sequence stars are pretty well modeled by $n = 3$ polytropes. That is, the run of temperature and density in the star roughly follows

$$T \propto \rho^{1/3} .$$

So the value $n = 3$ is a good one to keep in mind as we proceed, although we will meet other values later.

Hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

Mass:

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

Thus, eliminating $M(r)$,

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\cdot\frac{dP}{dr}\right] = -4\pi G\rho$$

$$P = K\rho^{1+\frac{1}{n}}$$

Change to *dimensionless units*:

ξ = dimensionless length, $\dfrac{r}{a}$, where a = some scale distance

ϕ = dimensionless density function (actually, very *temperature-like*)

$\rho = \rho_c\phi^n$, where ρ_c = central density, and $\phi(0) = 1$

$$P = K\rho_c^{1+\frac{1}{n}}\phi^{n+1}$$

Substitute for P:

$$\frac{1}{a^2\xi^2}\cdot\frac{1}{a}\frac{d}{d\xi}\left[\frac{a^2\xi^2}{\rho_c\phi^n}\cdot\frac{1}{a}\frac{d}{d\xi}\left(K\rho_c^{1+\frac{1}{n}}\phi^{1+n}\right)\right] = -4\pi G\rho_c\phi^n$$

$$\frac{1}{a^2\xi^2}\frac{d}{d\xi}\left[\frac{\xi^2}{\rho_c\phi^n}\cdot K\rho_c^{1+\frac{1}{n}}(1+n)\phi^n\frac{d\phi}{d\xi}\right] = -4\pi G\rho_c\phi^n$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n\left(\frac{4\pi Ga^2}{(1+n)\rho_c^{\frac{1-n}{n}}K}\right)$$

We now can see that an "inspired" choice for the length scale a would be

$$a^2 = \frac{(1+n)\rho_c^{\frac{1-n}{n}}K}{4\pi G}$$

and we obtain the canonical form of the *Lane–Emden equation*:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n \quad .$$

It is a common procedure in physics to convert equations to "mathematical" form like this!

5.2.5 Boundary conditions and Lane–Emden functions

The above Lane–Emden equation can only be integrated numerically. Conceptually, we first rewrite the equation as the equivalent form

$$\phi'' = -\left[\frac{2}{\xi}\phi' + (\phi)^n\right]$$

where prime denotes $d/d\xi$. Then starting at $\xi = 0$ with known *boundary conditions* on $\phi(0)$ and $\phi'(0)$, we start "stepping" in ξ. For each (theoretically infinitesimal) step, we update ϕ by its Taylor series

$$\phi(\xi + d\xi) = \phi(\xi) + \phi'(\xi)\, d\xi$$

and also update ϕ' by its Taylor series

$$\begin{aligned}\phi'(\xi + d\xi) &= \phi'(\xi) + \phi''(\xi)\, d\xi \\ &= \phi'(\xi) - \left(\frac{2}{\xi}\phi' + \phi^n\right) d\xi\ .\end{aligned}$$

In practice there are better numerical methods than this, but they reduce to this one conceptually.

What are the boundary values $\phi(0)$ and $\phi'(0)$? First, $\phi(0) = 1$ by definition of ρ_c (see above). Second, we know that $dP/dr = 0$ at $r = 0$, since GM/r^2 (local acceleration of gravity) goes to zero there. Thus from

$$P \propto \rho^{1+\frac{1}{n}} \propto \phi^{n+1}$$

we get

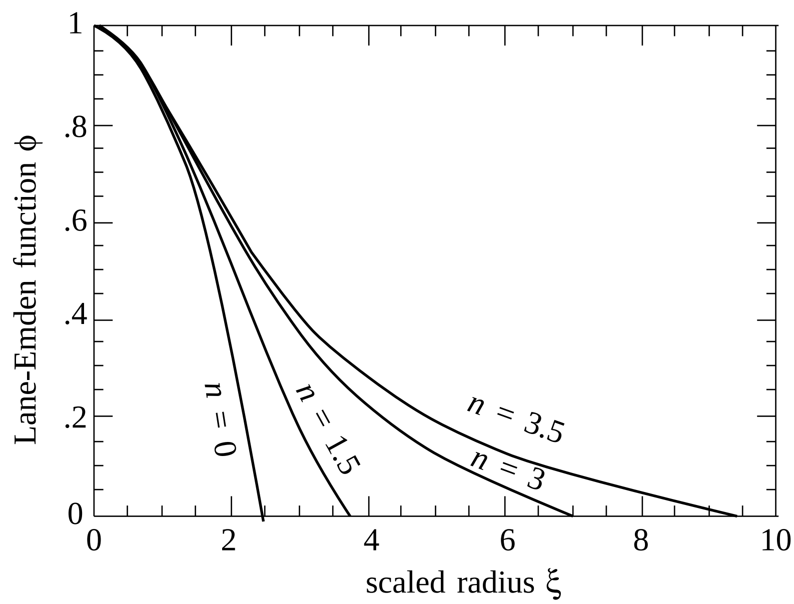
$$0 = \frac{dP}{dr} \propto \left(1 + \frac{1}{n}\right) \rho^{\frac{1}{n}} \frac{d\rho}{dr} \propto (n+1)\phi^n \frac{d\phi}{dr} \propto (n+1)\phi^n \phi'\ .$$

Since $\phi^n(0)$ is finite (actually $= 1$), $\phi'(0)$ must vanish.

With these boundary conditions, you might enjoy deriving the first terms in the power series expansion for $\phi(\xi)$, namely

$$\phi = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots \ .$$

Numerical integration of the kind just described can give these so-called *Lane-Emden functions* to any desired accuracy. Here is a graph for $n = 0, 1.5, 3.0$, and 3.5.



5.2.6 Physical properties of polytropes

For $n < 5$, the Lane-Emden function ϕ goes to zero at a finite value of ξ (and therefore r) which is called ξ_1. This is the surface of the star! The stellar radius in physical units is therefore

$$R = a\xi_1 = \sqrt{\frac{(1+n)\rho_c^{\frac{1-n}{n}} K}{4\pi G}}\xi_1 \ .$$

For the mass $M(r)$, we have

$$M(r) = \int_0^r 4\pi r^2 \rho \, dr = 4\pi a^3 \rho_c \int_0^{\xi=r/a} \xi^2 \phi^n d\xi \ .$$

However (here is a great trick!) if we multiply the Lane-Emden equation by ξ^2 and integrate from 0 to ξ we get

$$\xi^2 \frac{d\phi}{d\xi} = -\int_0^\xi \xi^2 \phi^n d\xi \ .$$

So we can immediately read off

$$\begin{aligned} M(r) &= -4\pi a^3 \rho_c \left(\xi^2 \frac{d\phi}{d\xi} \right)_{\xi=r/a} \\ &= -\frac{1}{\sqrt{4\pi}} \left[\frac{(n+1)K}{G} \right]^{3/2} \rho_c^{\frac{3-n}{2n}} \left(\xi^2 \frac{d\phi}{d\xi} \right)_{\xi=r/a} \ . \end{aligned}$$

The total mass is obtained by setting $\xi = \xi_1$ (stellar surface). To get physical masses and radii we thus need tabulated numerical values for ξ_1 and $-\xi_1^2 (d\phi/d\xi)_{\xi=\xi_1}$ (you could in principle read these off the above graph of the Lane-Emden functions)

n	ξ_1	$-\xi_1^2 \left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47

You might wonder what happens if you integrate the Lane-Emden equation beyond its first root ξ_1? Don't even think about it! If ϕ is negative then the density is negative, which is completely unphysical. The equation is perfectly "happy" to be truncated at $\xi = \xi_1$, since $\phi = 0$ implies $P = 0$, which is the correct surface boundary condition.

Recapping, the polytrope stellar model gives 2 algebraic relations among the 4 quantities K, ρ_c, M, and R. Nowadays we might view M and K as the independent variables (K, which puts a scale on how hot the star is, deriving from nuclear theory) and use the model to determine ρ_c and R. Historically, before nuclear processes were understood, people used *measured* values for M and R and then derived ρ_c and K, which were otherwise unknown. From either viewpoint, once these 4 quantities are known, we can go on to calculate how all quantities of interest vary with radius, for example by using the graphs of the Lane-Emden functions given above. Also, one can easily derive relations like

$$\frac{\overline{\rho}}{\rho_{\text{center}}} = -\frac{3}{\xi_1}\frac{d\phi}{d\xi}\bigg|_{\xi=\xi_1} \qquad \text{mean density of polytrope}$$

$$P_{\text{center}} = \frac{1}{4\pi(n+1)(\phi_1')^2} \times \frac{GM^2}{R^4} \qquad \text{central pressure}$$

$$T_{\text{center}} = \frac{\mu_{\text{center}}\, m_p}{k} \cdot \frac{P_{\text{center}}}{\rho_{\text{center}}} \qquad \text{central temperature (using the perfect gas law)}$$

<u>Example</u>: $n = 3$ polytrope: $M = 1\ M_\odot$; $R = 1\ R_\odot$ (use these measured values to get ρ_c and K).

From the numerical solution, $\xi_1 = 6.90$, $-\xi_1^2\phi_1' = 2.02$, and $\phi_1' = -0.0424$. Thus,

$$a^3\rho_c = -\frac{M}{4\pi\xi_1^2\phi_1'} = 7.9 \times 10^{31} \text{ g} \quad ,$$

and the scale length

$$a = 1.01 \times 10^{10} \text{ cm} \quad .$$

Therefore, if the Sun can be represented as a $n = 3$ polytrope, it has

central density, $\rho_c = 76.7 \text{ g cm}^{-3}$

mean density $= 0.0184\rho_c = 1.41 \text{ g cm}^{-3}$

central pressure, $P_c = 1.25 \times 10^{17} \text{ dyne cm}^{-2}$

central temperature, $T_c = 1.97 \times 10^7 \mu \text{ K}$

We also get the constant $K = 3.85 \times 10^{14}$ cgs (in $P = K\rho^{4/3}$).

5.3 Specific cases of polytropic models

5.3.1 Adiabatic indices for a perfect gas

If you compress a gas element whose thermal conductivity is small, so that there is no heat conducted into or out of the element, then its pressure is said to increase *adiabatically.* In this case we can use the 1st Law of Thermodynamics to get a relation between P and ρ for the gas element.

Consider a volume V containing N particles. Then the total energy E is given by the rule "$1/2kT$ per degree of freedom," namely

$$E = \frac{\beta}{2}NkT$$

where β is the number of degrees of freedom. For a monatomic, fully ionized, gas (the usual case in stars) we have $\beta = 3$ corresponding to x, y, and z translational motions.

The perfect gas law involves only the number density of particles N/V, not β, and is

$$P = \frac{N}{V}kT$$

which gives a relation between P and E for perfect gasses,

$$E = \frac{\beta}{2}PV \ .$$

Now the 1st Law of Thermodynamics, which is really just conservation of energy, says that when you squeeze a gas from volume V to volume $V - dV$ (smaller volume), the increase in its internal energy is just the work you have done squeezing it:

$$dE = -PdV \ .$$

(The minus sign is because the internal energy increases with decreasing volume.) Combining the last two equations (taking a differential of the first):

$$\begin{aligned}
\frac{\beta}{2}(PdV + VdP) &= -PdV \\
\frac{\beta}{2}VdP &= -\left(1 + \frac{\beta}{2}\right)PdV \\
\frac{dP}{P} &= -\left(\frac{2+\beta}{\beta}\right)\frac{dV}{V} \ .
\end{aligned}$$

Integrating, we get

$$P = \text{ constant} \times V^{-\left(1+\frac{2}{\beta}\right)} \ .$$

But since the density of a fixed quantity of gas (N particles) varies inversely with its volume, this is just

$$P \propto \rho^{\left(1+\frac{2}{\beta}\right)}$$

which is polytropic with index $n = \beta/2$. The most common case, $\beta = 3$, gives $n = 1.5, P \propto \rho^{5/3}$. Notice that as the number of degrees of freedom

β increases, the polytropic index increases. In fact, the limiting case of an *isothermal* gas ($P \propto \rho T$, T constant, so $P \propto \rho$) corresponds to $\beta \to \infty$. This is because one can view the work compressing the gas as being spread over an infinite number β of "internal" degrees of freedom, resulting in no increase in temperature.

5.3.2 Fully convective stars

Convection is the buoyancy-driven process of dynamical circulation that carries heat upward in gas or liquid in a gravitational field. You see it when you heat a pot of water on the stove, or when the Sun heats the ground (and hence the nearby air) on the Earth. In essence convection is nothing more than "hot air rises and cool air sinks."

In general, convection transports heat much faster than conduction does. Thus, a fluid element in the convective flow is very nearly adiabatically compressed (as it sinks) or decompressed (as it rises). Since convection is also generally *turbulent*, the mixing of different fluid elements is efficient. Thus, a gas in turbulent convection is quite accurately all *on the same adiabat.* That is, if it is monatomic and fully ionized ($\beta = 3$, above), it satisfies

$$P = \text{ constant} \times \rho^{5/3}$$

where all fluid elements have the same constant. This is just what is needed for the validity of a polytropic model with $n = 3/2$.

The Sun is not fully convective; most of it is *stably stratified* with the deeper, denser material being on a "lower adiabat" (smaller value of constant in above equation). That is why the run of density and pressure in the Sun is better described by $P \propto \rho^{4/3}$ (polytropic index $n = 3$) than by $P \propto \rho^{5/3}$

(polytropic index $n = 1.5$), even though the material in the Sun *is* monatomic and fully ionized. The Sun has an outer convective envelope only in the last 1/6 or so of its radius.

Low mass stars, $\lesssim 0.3 M_\odot$, are almost completely convective, so they are good $n = 1.5$ polytropes. Also, stars of all masses go through an initial convective phase (called the Hayashi phase) before they settle down to the main sequence. This phase can typically last several million years. It, too, is well described by the Lane-Emden equations for an $n = 1.5$ polytrope.

5.3.3 Equation of state for degenerate matter

Degenerate matter is matter whose resistance to compression comes not from thermal, the kinetic energy of its particles, but rather from the fact that its electrons (or, more generally, fermions) obey the Pauli exclusion principle and can't occupy the same quantum state. Thus, degenerate matter resists compression even at zero temperature.

Normal terrestrial solids and liquids (rocks, water, etc.) are, roughly speaking, degenerate in this sense, although the fact that their electrons are bound into atoms is an additional complication. In astrophysics, cold (or nearly cold) dead stars known as white dwarf stars are composed of degenerate matter (nucleii plus electrons). Neutron stars, composed of virtually 100% neutrons, are also degenerate. One says that these objects are "supported by degeneracy pressure."

Since temperature doesn't enter into it, degenerate matter is barytropic, with pressure a function of density only, $P = P(\rho)$. Let's calculate this equation of state in detail.

In Section 5.3.1, when we applied the 1$^{\text{st}}$ Law of Thermodynamics, the temperature T entered only as an intermediate step in getting a relation between P and E/V (energy per volume), namely

$$P = \frac{2}{3}\frac{E}{V} \qquad \text{(perfect gas)}.$$

Suppose we had gotten some different constant γ'

$$P = \gamma' \frac{E}{V} \qquad \text{(generalization)}.$$

Then, using this, plus the 1st Law:

$$\begin{aligned} -PdV = dE &= \frac{1}{\gamma'}(PdV + VdP) \\ \left(1 + \frac{1}{\gamma'}\right)\frac{dV}{V} &= -\frac{1}{\gamma'}\frac{dP}{P} \, . \end{aligned}$$

Integrating gives

$$P = \text{constant} \times V^{-(\gamma'+1)} \propto \rho^{\gamma'+1} \, .$$

So we see that we get a polytropic equation of state

$$P \propto \rho^{\gamma} \equiv \rho^{1+\frac{1}{h}}$$

with

$$\gamma' = \gamma - 1 = \frac{1}{n} \, .$$

In other words, a polytropic gas $P \propto \rho^{\gamma}$ has, in general,

$$PV = (\gamma - 1)E \, .$$

Now back to degenerate electrons. In our discussion of quantum phase space density in Section 3.5.2, we already saw that fermions, because of the Pauli exclusion principle, have a maximum phase space density of 1 per quantum unit h^3 of phase space. Actually it's 1 for each so-called spin state, just as, when for photons, we counted each polarization separately. Electrons have two spin states, called $+\frac{1}{2}$ and $-\frac{1}{2}$.

Recall that phase space volume is the product of actual space volume and momentum space volume $d^3\boldsymbol{x}d^3\boldsymbol{p}$. Because of the limitation on phase space density, if we force some electrons into a smaller volume $d^3\boldsymbol{x}$, they must push out into a larger momentum volume $d^3\boldsymbol{p}$. The lowest energy state of an electron gas (the state it will take on in practice at zero or low temperature) is when the electrons fill all momentum states in a sphere out to some radius p_F in momentum space (called the Fermi momentum) and none beyond. Thus, for N electrons in a volume V we have the phase space density

$$\mathcal{N} = \frac{N}{V\left(\frac{4}{3}\pi p_F^3\right)} = \frac{2}{h^3} .$$

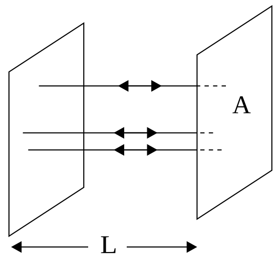
Since N/V is n_e, the number density of electrons, we have

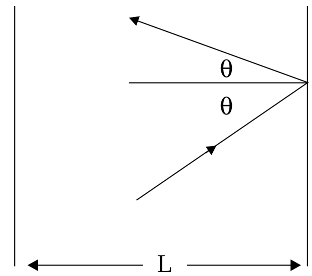
$$p_F^3 = \frac{3}{8\pi} h^3 n_e .$$

Now to get the energy per volume, E/V, we integrate over the density distribution in momentum space, putting in $E_e = p^2/(2m_e)$ as the energy of each electron. (Note that this is valid for *non-relativistic electrons*)

$$\frac{E}{V} = \int E_e \mathcal{N} d^3\boldsymbol{p} = \int_0^{p_F} \frac{p^2}{2m_e}\frac{2}{h^3}4\pi p^2 dp = \frac{4\pi}{5}\frac{p_F^5}{h^3 m_e} .$$

Now what about the relation between pressure P and energy density E/V? Actually, degeneracy never comes into this. It is just the same calculation we did in Section 3.3.4, but for a nonrelativistic particle moving at velocity V. Go back and look at the equations around these figures





and you should then understand the equations,

$$\Delta P = \frac{1}{3}\left[\frac{2p}{A(2L/v_e)}\right] = \frac{1}{3}\left[\frac{2p}{A(2L/[p/m_e])}\right] = \frac{1}{3}\frac{p^2}{ALm_e} = \frac{2}{3}\frac{\Delta E}{V}$$

or

$$P = \frac{2}{3}\frac{E}{V} \,.$$

So from the previous discussion of γ' we know right away that $P \propto \rho^{5/3}$. Another way to verify this and also get the constant in the relationship, is by substituting the above expression for E/V in terms of p_F^5, and also substituting the earlier expression for p_F^3 in terms of n_e. The result is the amazing universal formula

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e}n_e^{5/3} \qquad \text{(non-relativistic electrons).}$$

Something different happens if the electron gas is so compressed that the Fermi momentum p_F starts approaching $m_e c$ (i.e. the electron velocity starts approaching c). Then, the above assumptions $E_e = p^2/(2m_e)$ and $v_e = p/m_e$ start breaking down! In the extreme *relativistic* limit, the above calculation of ΔP goes over to being exactly the same as we previously did for photons, so

$$P = \frac{1}{3}\frac{E}{V} \qquad \Rightarrow \qquad \gamma' = \frac{1}{3} \qquad \Rightarrow \qquad \gamma = \frac{4}{3} \qquad \Rightarrow \qquad P \propto \rho^{4/3} \,.$$

The calculation of E/V now uses $E_e = pc$, and is

$$\frac{E}{V} = \int E_e \mathcal{N} d^3\boldsymbol{p} = \int_0^{p_F} pc\frac{2}{h^3}4\pi p^2 dp = \frac{2\pi c}{h^3}p_F^4 \ .$$

So the same two substitutions as before give

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc\, n_e^{4/3} \qquad\qquad \text{(relativistic electrons).}$$

The crossover between these two limits can be roughly taken to be where they are equal,

$$\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc\, n_e^{4/3} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e} n_e^{5/3}$$
$$\Rightarrow n_e = \frac{4}{3}\left(\frac{2}{5}\cdot\frac{h}{m_e c}\right)^{-3} \ .$$

The length $h/m_e c = 2.42 \times 10^{-10}\,\text{cm}$ is called the Compton wavelength of the electron. An electron forced into a box smaller than about 2/5 Compton wavelengths on a side, we see, becomes relativistic. This is of course a consequence of the Heisenberg uncertainty principle relating positional localization to momentum *de*-localization.

A final detail for our astrophysical applications is to relate n_e, the electron density, to ρ, the total mass density. Except in the case of hydrogen (which is not present in white dwarf stars anyway) each electron is accompanied by exactly one proton *and* by very nearly one neutron. Thus

$$\rho = n_e(m_e + m_p + m_n) \cong 2n_e m_p \ .$$

We can then substitute $n_e = \rho/(2m_p)$ in the above equations for P, getting the actual equation state $P = P(\rho)$. More precisely we might define μ_e as the mean molecular weight per electron, and use $\rho \equiv \mu_e m_p n_e$, with $\mu_e = 1$

for hydrogen, ≈ 2 for everything else. If a gas contains a mass fraction X of hydrogen,

$$X \equiv \rho_H / \rho$$

then you can readily work out that

$$\mu_e \approx \frac{2}{1+X} \,.$$

Elsewhere, you might see μ (not μ_e), which is defined as the mean molecular weight *per particle* (not per electron). If $Y \equiv \rho_{\rm He}/\rho$ is the helium mass fraction, and $Z = (1 - X - Y)$ is the mass fraction of everything heavier than He, then a good approximation is

$$\mu \approx \frac{2}{1 + 3X + \frac{1}{2}Y} \,.$$

Here is a handy table of values for mixtures of H and He:

Pure H gas:	$\mu_e = 1\,,$	$\mu = \frac{1}{2}$		
Hydrogenless gas:	$\mu_e = 2\,,$	$\mu = \dfrac{2}{1+\frac{1}{2}Y}$		
"Cosmic gas":	$X = 0.74\,,$	$Y = 0.26$	$\Rightarrow$	$\mu_e = 1.15\,,\quad \mu = 0.60$

5.3.4 White dwarf stars

White dwarfs are stars supported entirely by electron degeneracy pressure, so we can immediately apply the results of the previous section:

$$\begin{aligned}
P_e &= \left[\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e m_p^{5/3} \mu_e^{5/3}}\right] \rho^{5/3} \qquad \text{(nonrelativistic)} \\
P_e &= \left[\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{m_p^{4/3} \mu_e^{4/3}}\right] \rho^{4/3} \qquad \text{(extreme relativistic).}
\end{aligned}$$

Write these as

$$P_e = K_e \rho^{1+\frac{1}{n}} \quad ,$$

with $n = \frac{3}{2}$ for the nonrelativistic case and $n = 3$ for the very relativistic case. Then, from polytrope theory (Section 5.2.6), a white dwarf radius and mass are

$$R = \xi_1 \left[\frac{(n+1)K_e}{4\pi G} \right]^{1/2} \rho_c^{(1-n)/(2n)}$$

$$M = -\xi_1^2 \phi_1' \, . \, 4\pi \left[\frac{(n+1)K_e}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/(2n)}$$

Nonrelativistic White Dwarf

Take $n = \frac{3}{2}$; then

$$\begin{aligned} \xi_1 &= 3.654 \\ -\xi_1^2 \phi_1' &= 2.714 \\ R &= \left(1.122 \times 10^4 \text{ km}\right) \left(\rho_c / 10^6 \text{ g cm}^{-3}\right)^{-1/6} (\mu_e/2)^{-5/6} \\ M &= (0.4964 \; M_\odot) \left(\rho_c / 10^6 \text{ g cm}^{-3}\right)^{1/2} (\mu_e/2)^{-5/2} \; . \end{aligned}$$

Eliminate the unknown central density ρ_c:

$$M = (0.7011 \; M_\odot) \left(R/10^4 \text{ km}\right)^{-3} (\mu_e/2)^{-5} \; .$$

So $M \propto R^{-3}$, with $\rho_c \sim 10^7$ g cm^{-3} when $M \sim 1.5 \; M_\odot$, and the more massive a white dwarf is, the smaller it is!

Relativistic White Dwarf

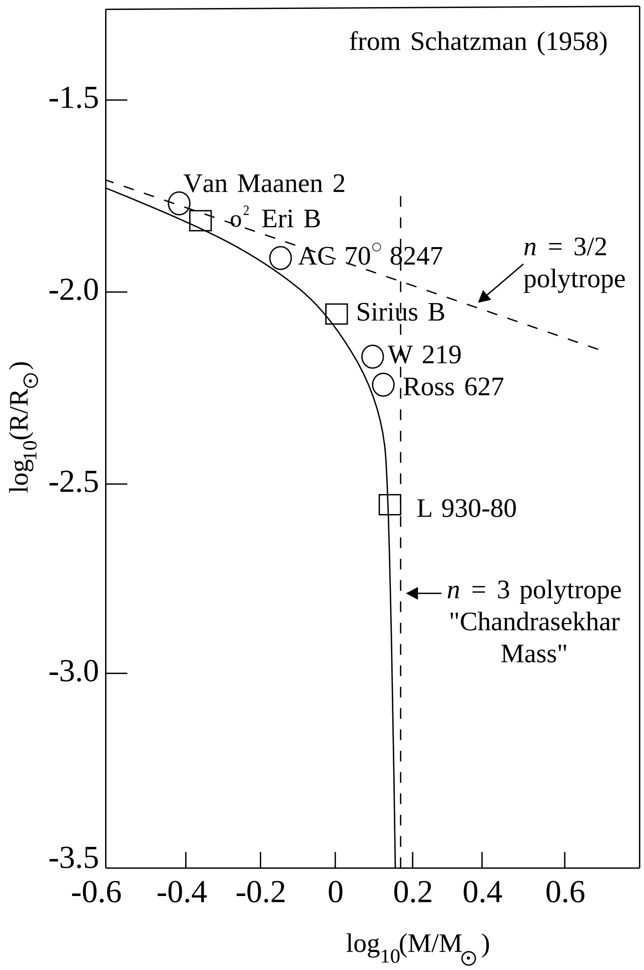
Take $n = 3$; then

$$\xi_1 = 6.897$$

$$
\begin{aligned}
-\xi_1^2\phi_1' &= 2.018 \\
R &= \left(3.347 \times 10^4 \text{ km}\right)\left(\rho_c/10^6 \text{ g cm}^{-3}\right)^{-1/3}(\mu_e/2)^{-2/3} \\
M &= (1.457\ M_\odot)\,(2/\mu_e)^2\ .
\end{aligned}
$$

Isn't it peculiar that M is independent of ρ_c! It has a fixed value, $1.457 M_\odot$ (if $\mu_e = 2$), that is called the "Chandrasekhar mass," $M_{\rm Ch}$. What does this mean? Recall that the extreme relativistic case is a *limiting* case as $\rho_c \to \infty$. Thus, the meaning is that as $\rho_c \to \infty$ we get driven to the relativistic case with $M \to M_{\rm Ch}$. That is the maximum mass that electron degeneracy pressure can possibly support. For higher masses there are simply *no* solutions!

Another way of understanding this is to look at the mass-radius diagram, plotting both the polytropes $n = 3/2$ and $n = 3$, and also a more complicated model that goes smoothly from the non-relativistic to extreme relativistic limits:



So, over this range the radius for a given mass decreases; and at $R = 0$ ($M = M_{\rm Ch}$) the relativistic electrons can *no longer* support the star. The points plotted on the curve are observational measurements of actual white dwarf stars, and demonstrate that our theory is basically correct, even accurate quantitatively!

Note that $\mu_e = 2$ is appropriate because $X \simeq 0$: the star must have burned all its hydrogen en route (used up all nuclear fuel).

In terms of fundamental constants, we can write

$$M_{\rm Ch} = 3.10 \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2 \mu_e^2} \quad ,$$

where

$$\left(\frac{\hbar c}{G}\right)^{1/2} \equiv m_{\rm Planck} \quad .$$

This so-called Planck mass has the numerical value 0.22 μg. So $M_{\rm Ch}$ depends only on $m_{\rm Planck}$ and m_p, even though it is *electrons* that support the star. Nowhere does m_e enter the formula. In the relativistic limit, the rest mass *does not* enter the $E/(PV)$ relation. Thus, m_e does not enter into the calculation of electron pressure.

5.3.5 Stellar structure virial theorem

We have seen that as $\gamma \to 4/3$ from above (e.g. from $5/3 \to 4/3$ for relativistic electrons), something "goes wrong" with the stellar structure equations. We might ask as a point of principle why we can't build a star out of material with an even *smaller* γ. For example, a molecule a like NH_4 has not only the 6 obvious degrees of freedom (3 translational and 3 rotational) but also 9 *vibrational* modes. So a star made of convecting ammonia would be a polytrope with

$$\gamma = 1 + \frac{2}{15} < 4/3 \ .$$

(This is of course fanciful, since stars are always so hot that molecules are destroyed. But there is a point of principle to understand here.)

We can use thermodynamic results and the pressure equation to deduce a virial theorem relating to stellar structure. Start from the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\,\frac{GM(r)\,\rho(r)}{r^2}$$

and define $V(r)$ = volume occupied by gas inside radius $r = \frac{4}{3}\pi r^3$, so that dV = volume of dr-shell, containing mass $\rho(r)\,4\pi r^2 dr$.

Multiply the pressure equation by $V(r)\,dr$:

$$V(r)\frac{dP}{dr}\,dr = -\,\frac{GM(r)\,\rho(r)}{r^2}\,V(r)\,dr \quad ,$$

i.e.,

$$V(r)\,dP = -GM(r)\,\rho(r)\,\frac{4}{3}\pi r^3 dr\,\frac{1}{r^2} = -\,\frac{1}{3}GM(r)\,dM(r)\,\frac{1}{r} \quad .$$

Integrate over the star

$$\int\limits_{r=0}^{r=R} V(r)\,dP = -\,\frac{1}{3}\int\limits_{r=0}^{r=R}\frac{GM(r)\,dM(r)}{r} = \frac{1}{3}U \quad ,$$

where U is the total gravitational potential energy. Hence, integrating by parts,

$$\frac{1}{3}U = [PV]_{r=0}^{r=R} - \int\limits_{r=0}^{r=R} P(r)\,dV \quad .$$

At $r = 0$, $V = 0$. At $r = R$, $P \simeq 0$. (The surface of a star is approximately a vacuum.) Therefore,

$$[PV]_{r=0}^{r=R} = 0 \quad .$$

Hence,

$$U + 3\int_{r=0}^{r=R} P\,dV = 0\,.$$

This is the most general form of the stellar structure virial theorem.

For an ideal gas, the energy per volume u is

$$u = \frac{P}{\gamma - 1} \quad .$$

(We formerly wrote this as $u = E/V$ when it was assumed constant over the volume V.) Thus,

$$\int P\,dV = (\gamma - 1)\int u\,dV = (\gamma - 1)\,E\,,$$

where E is the total internal energy of gas. Therefore,

$$U + 3(\gamma - 1)E = 0 \ .$$

Note that for a perfect, monatomic gas, $\gamma = \frac{5}{3}$, and the total internal energy/unit mass is just the total kinetic energy of the gas particles, i.e., $E \equiv T$. So $U + 3(\gamma - 1)E = 0$ is equivalent to $U + 2T = 0$, the same as self-gravitating particle virial theorem. (Theorems are consistent – fortunately!)

The total energy of a star is

$$E_{\rm tot} = E + U$$

so

$$\begin{aligned} E_{\rm tot} = E - 3(\gamma - 1)E &= -(3\gamma - 4)E \\ &= +\frac{(3\gamma - 4)}{3(\gamma - 1)} U \quad . \end{aligned}$$

We now see that

$$\begin{array}{llll} \text{if} & \gamma > \dfrac{4}{3} \ , & E_{\rm tot} < 0 & \text{(therefore, bound star)} \\ \text{if} & \gamma < \dfrac{4}{3} \ , & E_{\rm tot} > 0 & \text{(therefore, unbound “star”).} \end{array}$$

We see that stars are stable only if their adiabatic index γ exceeds $\frac{4}{3}$. Otherwise, they are unstable to converting their internal energy into expansion velocity — they blow themselves apart!

5.4 Beyond the Chandrasekhar mass

What happens as M approaches or exceeds $M_{\rm Ch}$? Further physical processes come into play as the central density, ρ_c, increases.

5.4.1 Inverse β decay

Normally, neutrons decay by $n \to p + e^- + \overline{\nu_e}$ with energy release ~ 1 MeV. Therefore, when the Fermi energy per electron becomes ~ 1 MeV, an *inverse* reaction $e^- + p \to n + \nu_e$ can go. As the radius decreases, the density rises and E_F rises until $E_F \to 1$ MeV. Then, the electrons disappear by combining with protons to produce neutrons. This means that the electron pressure drops, and the electrons become less able to support the star. The star collapses further and faster until all electrons combine with protons. Thus, the star becomes a mass of neutrons, a *neutron star*.

The collapse may go even further – there is a limiting mass for a neutron star, too. (We can calculate that in the same way as for electrons.)

5.4.2 Neutron stars and pulsars

Neutron stars are exactly like white dwarf stars in the theory of their structure, but with support from degenerate neutrons instead of electrons. Suppose these neutrons form a gas like the electrons (actually, they are thought to "solidify" or "liquify" in some parts of the star).

In the case of relativistic neutrons, the neutron degeneracy pressure is

$$P_n = \left[\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{m_p^{4/3}\mu_n^{4/3}}\right] \rho^{4/3} \quad ,$$

where μ_n is the mean molecular weight per neutron in a.m.u. $\simeq 1$ (since there is about 1 m_p per neutron). So we get the *same* $n = 3$ polytrope solution as electrons with

$$M = (1.457\ M_\odot)\left(\frac{2}{\mu_n}\right)^2 \approx 5.83\ M_\odot \quad ,$$

and the limiting mass of a neutron star is $\approx 5.83\ M_\odot$. The only change in the argument from the white dwarf case is to replace μ_e by μ_n.

Actually, this estimate is rather inaccurate: (1) because at this density nuclear forces (strong force) between neutrons are appreciable, and this aids gravity, thus decreasing the limiting mass; and (2) because the Newtonian gravitational potential $\phi_{\text{surface}}/c^2 \sim 1$, so Newtonian gravity is a poor approximation, and we should use General Relativity. The physics overall is rather uncertain, but the limiting mass of a neutron star is probably $\sim 3\ M_\odot$. Until we know the equation of state of a neutron fluid, this mass will remain uncertain.

What is the radius of a neutron star? Neutron stars are denser than white dwarf stars. In the limit, the central density should correspond to the neutrons almost touching. The neutron–neutron separation is then ~ 1 fm $\approx 10^{-13}$ cm, so the neutron number density $\sim 10^{39}$ cm^{-3}, and $\rho \sim 2 \times 10^{15}$ g cm^{-3}. Hence, if $M \sim 1\ M_\odot$, $R \sim 10$ km.

We *know* that neutron stars exist because of evidence of pulsars. (We know that white dwarfs exist, too, because we can see them optically; e.g., Sirius B.) Pulsars are distinguished by the regular arrival of radio pulses separated by intervals of a few ms (0.5 ms = shortest) to a few seconds (~ 5 sec = longest).

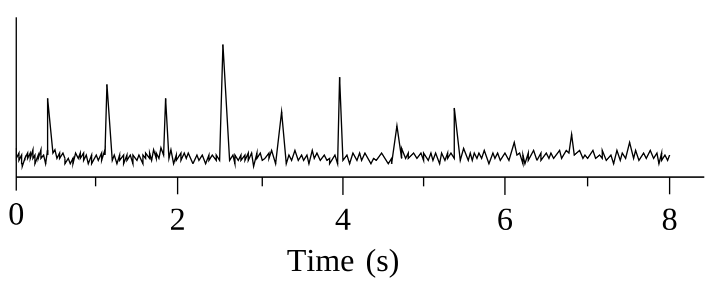
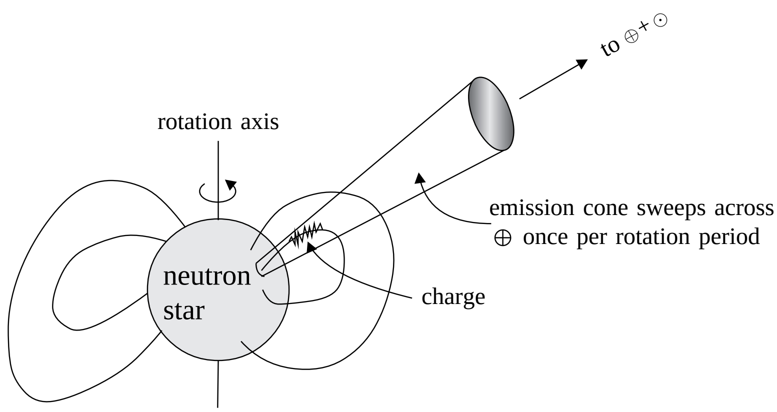


Chart record of individual pulses from one of the first pulsars discovered, PSF 0329 + 54. They were recorded at a frequency of 410 MHz and with an instrumental time constant of 20 ms. The pulses occur at regular intervals of about 0.714 s.

The period is quite regular (but slowly decreasing, due to "spindown" of the neutron star). The flux and pulse shape are somewhat variable, but the long-time averages are stable. Some pulsars put out optical and x-ray pulses as well as radio pulses.

What makes pulsars pulse? The answer is thought to be that the pulses are from electrons trapped in the magnetic field of the rotating neutron star. The idea is that the pulses are thus emitted in a *cone* rotating with the neutron star. Suppose the period is P $(= 2\pi/\Omega)$. Then we can derive a limit to the neutron star radius r. At the equator, the centrifugal force $= \Omega^2 r$ per unit mass had better not exceed the gravitational force $= GM/r^2$ per unit mass.



Therefore, $\Omega^2 r < GM/r^2$ for stability. So for a given P, we require

$$r < \left(\frac{GMP^2}{4\pi^2}\right)^{1/3} \sim 1500 \left(\frac{P}{\text{sec}}\right)^{2/3} \quad \text{km} \quad .$$

Crab pulsar: $P = 33$ ms $r < 150$ km $\lesssim \frac{1}{10}\times$ white dwarf radius

Pulsar 1937+21: $P = 2$ ms $r < 24$ km $\lesssim \frac{1}{100}\times$ white dwarf radius

These objects can't be white dwarf stars (periods too short), and they can't be planets (or the rate of loss of energy would slow them too quickly), so we infer that they are neutron stars.

We see neutron star slow down at a rate that is consistent with energy loss by electromagnetic radiation due to rotating *magnetic dipole*. This requires very strong magnetic fields, $\sim 10^{12}$ Gauss (believed to be present).

5.4.3 Black holes

What happens at $M > 6\ M_{\odot}$ (certainly greater than limiting mass of a neutron star)? Then matter collapses to a black hole; the star vanishes inside its Schwarzschild radius r_s.

$$r_s = \frac{2GM}{c^2}$$

A classical derivation of r_s (equate escape velocity to the speed of light) is possible, but wrong: this is a result of General Relativity. Putting in numerical values gives

$$r_s = 3\left(\frac{M}{M_{\odot}}\right) \quad \text{km} .$$

This is not a lot smaller than a neutron star: neutron stars are *almost* too small and dense to support themselves. Nothing escapes from the region $r < r_s$; and the gravitational field at $r \sim r_s$ is so high that almost nothing gets out from there either; the strong field causes infall of gas.

This infalling gas heats up as it falls in and may be seen by the x-rays it emits. The direct evidence for black holes is meager: they are hard to see! We require unseen object of high density and $M \gtrsim 4\ M_{\odot}$ (theoretical neutron star limit) in a binary system. A few such objects are known. (The evidence for massive black holes, $10^6 - 10^8 M_{\odot}$, at the center of quasars is a lot more certain.)