# Computational Structural Dynamics

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# Objective

- Introduction to modeling
  - An example
  - Strain, Equilibrium
  - Solution
  - An easier solution
- Solution Methods
  - Finite Difference
  - Weighted Residual
  - Finite Element
  - Others

# Objective

- The reality of FE modeling
  - Using the finite element method
  - Developments in modeling
  - Simulations

The modeler:

*Reality is intractable.*

The experimentalist:

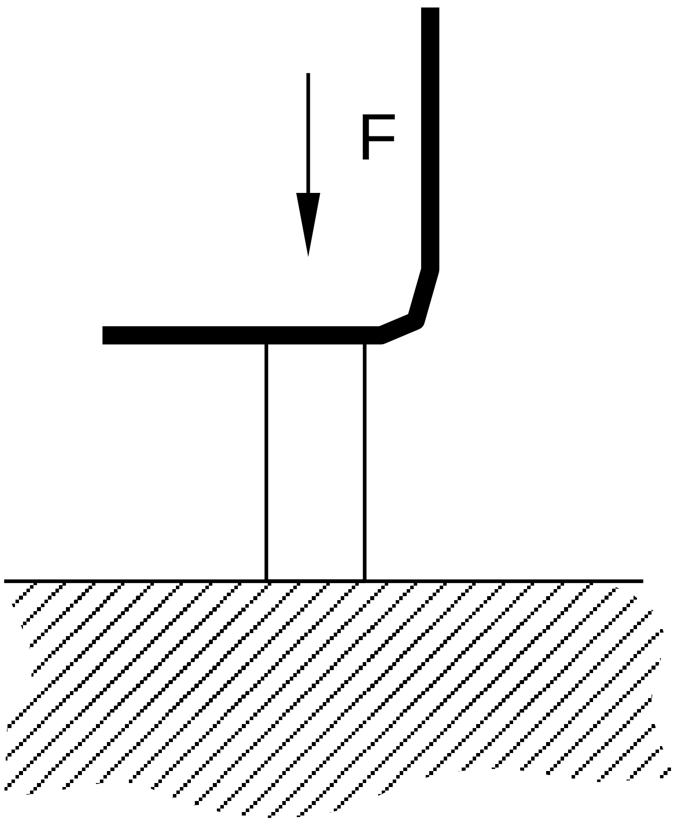
*It is what it is.*

# Will your chair hold you?

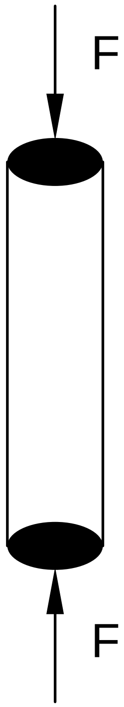
# Sit Down

It depends on which chapter in Shigley & Mischke we're studying.

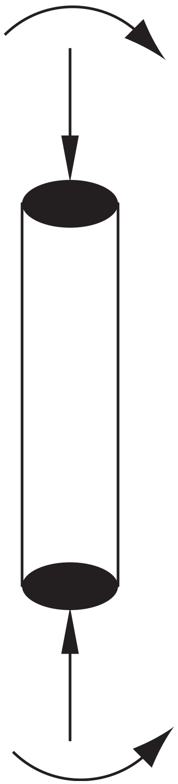
# Your Chair



# Your Chair

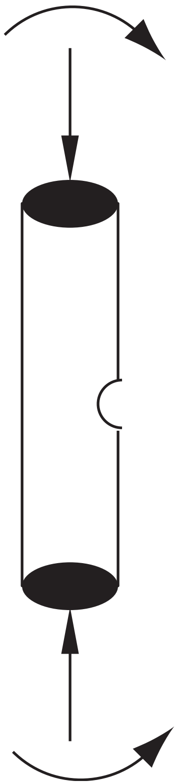



$$\sigma = \frac{F}{A}$$



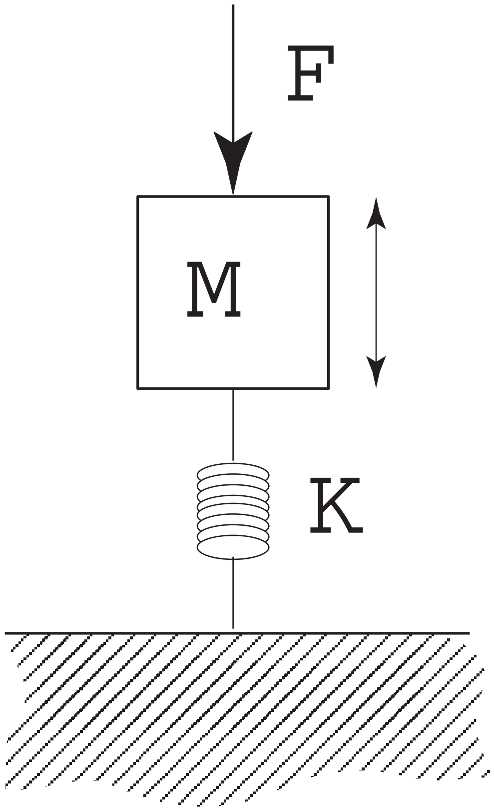
$$\sigma = \frac{F}{A} + \frac{Mc}{I}$$

# Your Chair



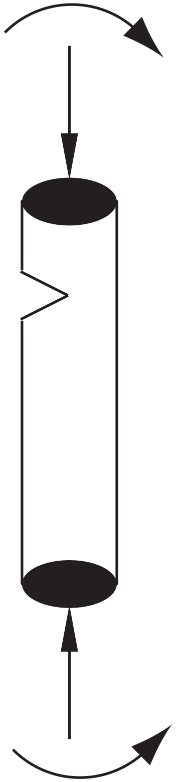
$$\sigma = \beta\left(\frac{F}{A} + \frac{Mc}{I}\right)$$

# Your Chair



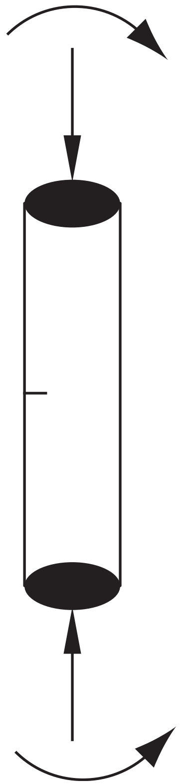

$$M \cdot \ddot{x} + K \cdot x = F(t)$$

# Your Chair

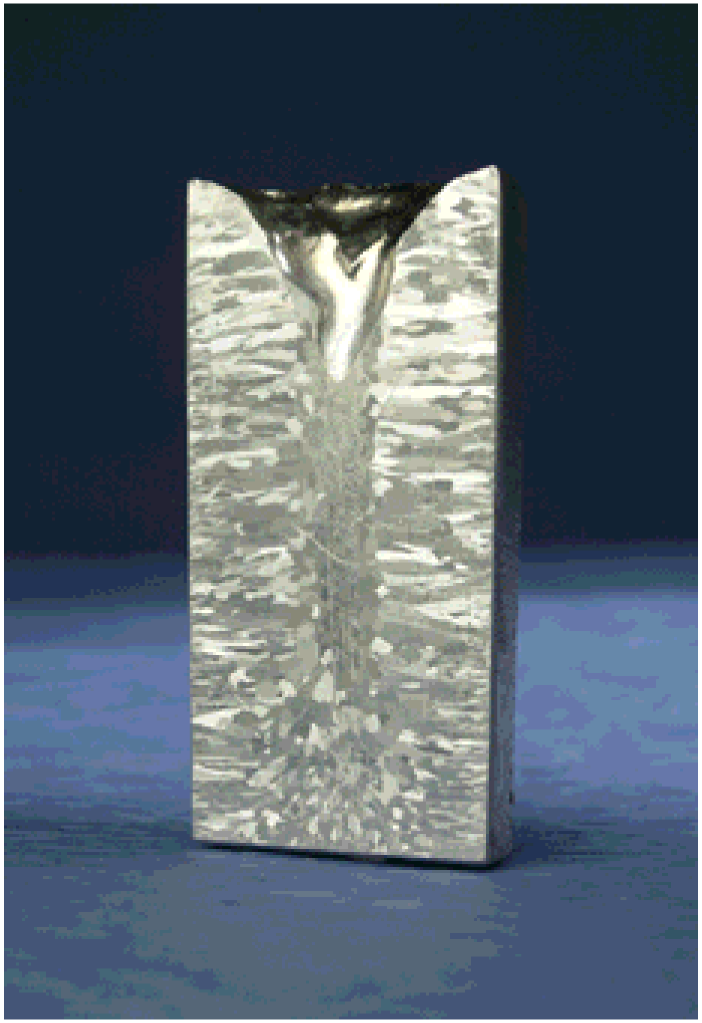


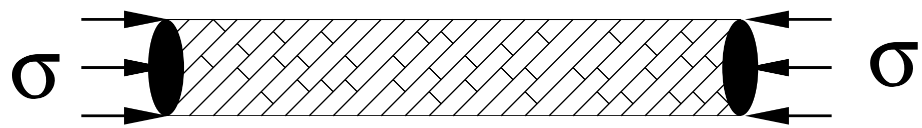
$$\sigma = K\sqrt{\pi \cdot a}$$

# Your Chair



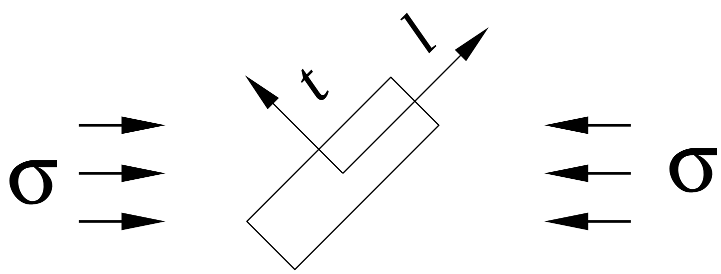
$$\frac{da}{dN} = C\left(\Delta K\right)^m$$






Macroscopically $\rightarrow$ $\sigma = E\,\epsilon$

Mesoscopically:




$$\boxed{\begin{array}{l} E_l \neq E_t \\ \{\sigma\} = [E]\,\{\epsilon\} \end{array}}$$

# Modeling your Chair

- Normal loads
- Bending loads
- Stress concentration
- Dynamics
- Fracture
- Fatigue
- Inhomogeneous

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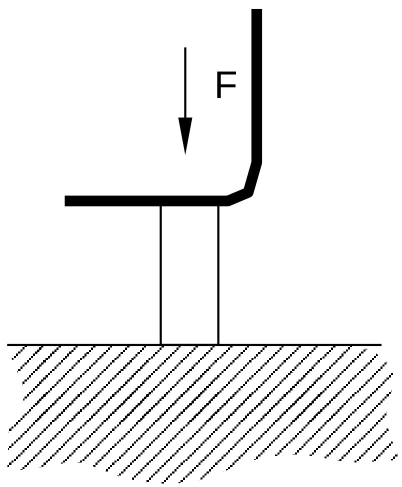
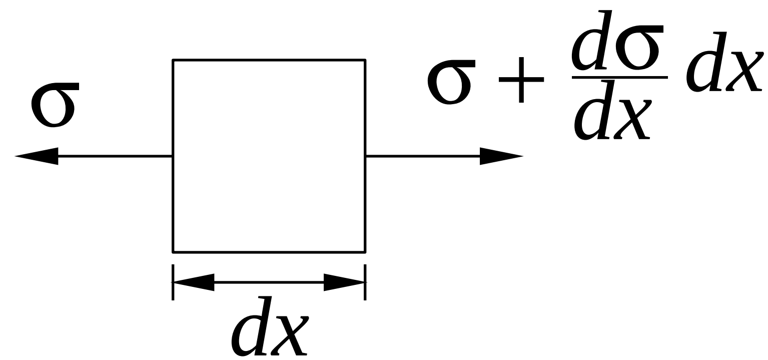
- Boundary conditions
- Geometry
- Important Physics
- "Complexity"

# Models

What decides the kind of model we need? $\longrightarrow$ It all depends on the results you want.




$$\sigma(t) = \frac{F(t)}{A}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

# What goes into a model? Physical Description

- Geometry
  - length of bar
  - location of discrete masses

- Material description
  - mass
  - constitutive model, E, $\sigma_y$
  - dissipation

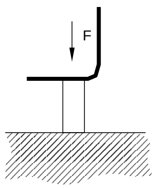
- Loading
  - $P = C$
  - $P = P(t)$

- Boundary Conditions
  - $\delta(L) = 0$
  - $\delta_y = 0$ or; $\epsilon_y = \nu\epsilon_x$

- Newtonian Mechanics
  - $\sum \vec{F} = m\,\vec{a}$
- Celestial Mechanics
  - $\vec{F}_{12} = \frac{G m_1 m_2}{r_{12}^2} \vec{r}_{12}$
- Energy Conservation
  - $1^{st}$ Law of Thermodynamics

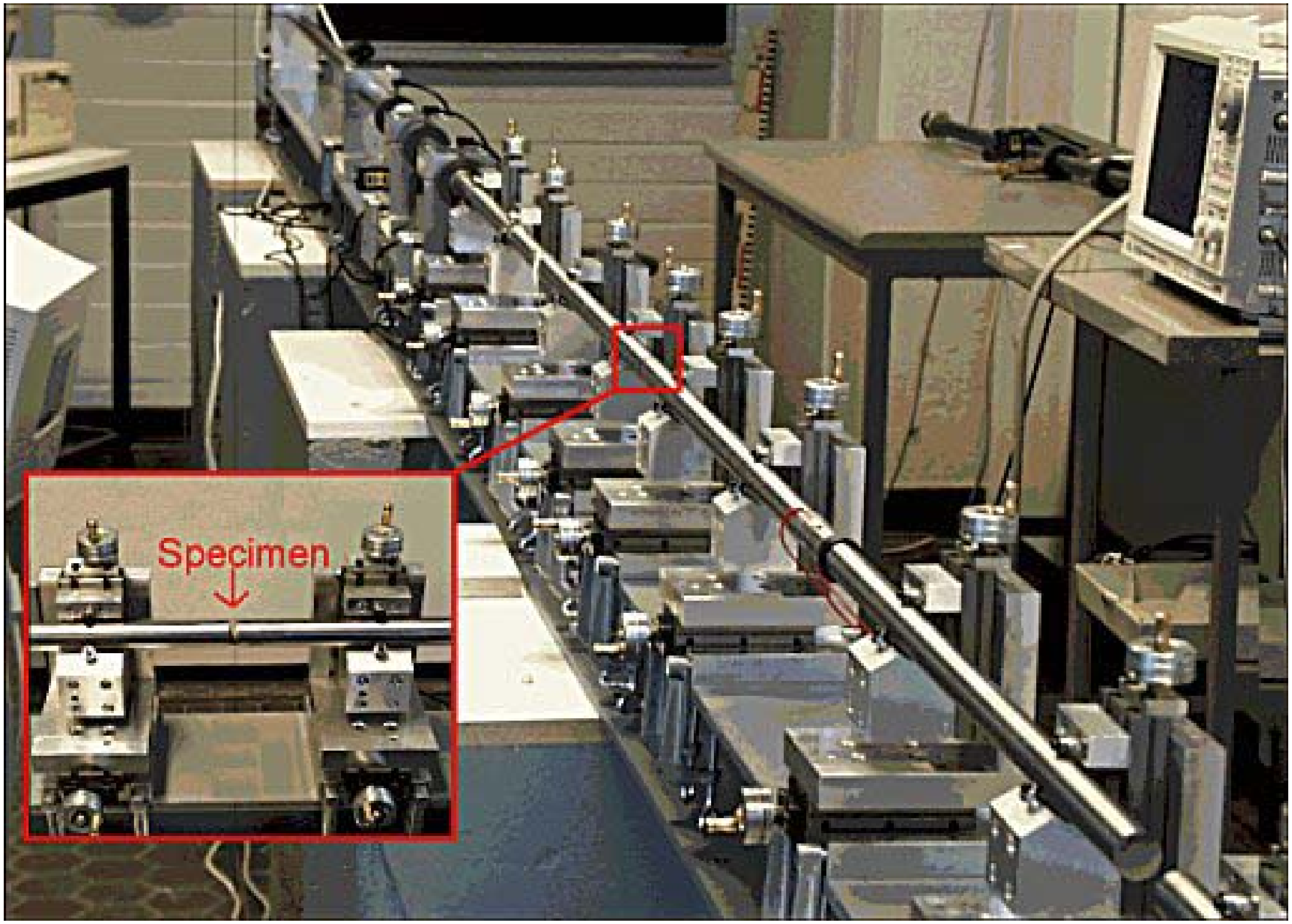
- Analytical
  - the exact solution to the *model*
- Approximate
  - $\epsilon = \frac{\partial u}{\partial x}$ only good for small $\epsilon$
- Numerical
  - often combines both approximation and analytical models

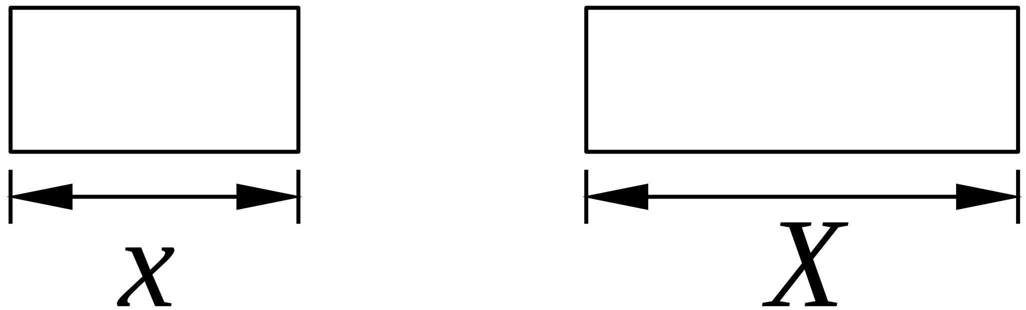
## What goes into a model? The Complexity

- Geometry
  - $l = l_0$
- Material
  - $\sigma = E\epsilon$
- Loading
  - $P(t) \rightarrow$ Fourier Series
- Boundary Conditions
  - $\delta(L) = 0$ 

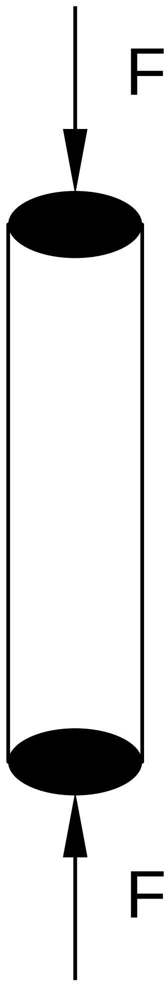
- Physics
  - Hooke's Law is a macroscopic model of inter-molecular forces.
- Solution
  - The solution method often grows to fit the solution power.

# Split Hopkinson Pressure Bar






$\lim_{x \to 0} \frac{X - x}{x} = \frac{\partial u}{\partial x}$ $\Rightarrow$ spatial rate of change of displacement




$$\delta(x) = \frac{F}{AE}x$$

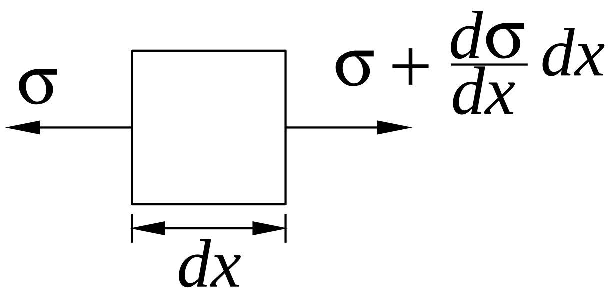
But, $\frac{F}{A} = \sigma$

so, $\delta(x) = \frac{\sigma}{E}x = \epsilon x$

$$\frac{\partial}{\partial x}\delta(x) = \frac{F}{AE} = \epsilon$$

# Split Hopkinson Pressure Bar

Equilibrium on a differential element




$$m = \rho V = \rho\, dx\, dy\, dz$$
$$\sum F = ma = (\rho\, dx\, dy\, dz)\; a$$
$$\sigma = E\epsilon = E\frac{\partial u}{\partial x}$$

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$$-\sigma\, dy\, dz + \left(\sigma + \frac{\partial \sigma}{\partial x}\, dx\right)\; dy\, dz = (\rho\, dx\, dy\, dz)\; \frac{\partial^2 u}{\partial t^2}$$

$$-\sigma\, dy\, dz + \sigma\, dy\, dz + \frac{\partial \sigma}{\partial x}\, dx\, dy\, dz = (\rho\, dx\, dy\, dz)\; \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x}\, dx\, dy\, dz = (\rho\, dx\, dy\, dz)\; \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x} = \frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right) = E\frac{\partial^2 u}{\partial x^2} = \rho\, \frac{\partial^2 u}{\partial t^2}$$

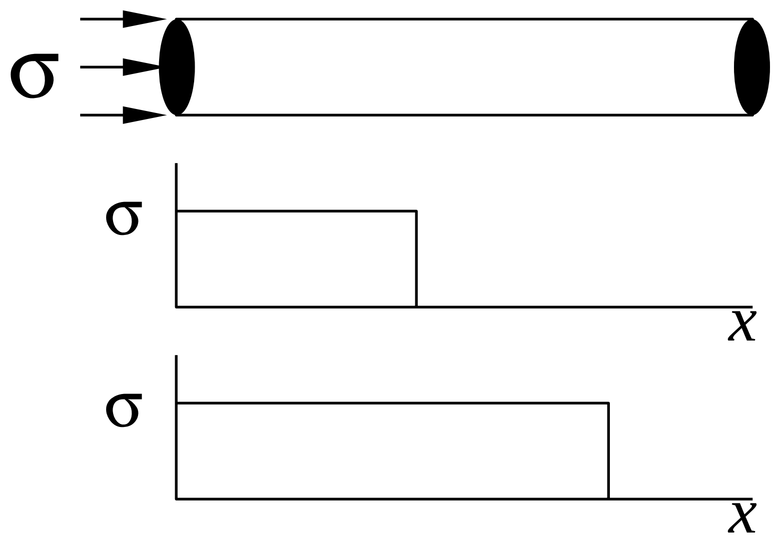
## The Wave Equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$c^2 = \frac{E}{\rho}$$

This is a differential equation that describes the *pointwise* motion of our rod.

# Solution by inspection




$$x > ct \qquad \sigma_x = 0$$

$$x \leq ct \qquad \sigma_x = \sigma$$

$$\epsilon = \frac{\sigma_x}{E} \qquad u = \int \epsilon \, dx$$

The strain displacement relationship:

In 1-D:

$$\lim_{x\to 0} \frac{X-x}{x} = \frac{\partial u}{\partial x}$$

In 3-D :

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right],$$

$i = 1,2,3; j = 1,2,3$

$$\epsilon_{11} = \left[\frac{\partial u_1}{\partial x_1}\right]; \quad \epsilon_{22} = \left[\frac{\partial u_2}{\partial x_2}\right]; \quad \epsilon_{33} = \left[\frac{\partial u_3}{\partial x_3}\right]$$

$$\epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right]$$

$$\epsilon_{23} = \epsilon_{32} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right]$$

$$\epsilon_{31} = \epsilon_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right]$$

Equilibrium $\longrightarrow$ equation of motion

In 1-D:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x} = \rho a$$

In 3-D :

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho a_i$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \rho a_1$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \rho a_2$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho a_3$$

# A solution

Constitutive behavior $\rightarrow$ Hooke's Law

In 1-D:

$$\epsilon_x = \frac{\sigma_x}{E}$$

$$\epsilon_y = \epsilon_z = -\nu\epsilon_x$$

In 3-D :

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

*or*

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\epsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$\epsilon_{22} = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$\epsilon_{33} = \frac{1+\nu}{E}\sigma_{33} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$\epsilon_{12} = \epsilon_{21} = \frac{1+\nu}{E}\sigma_{12}; \quad \epsilon_{23} = \epsilon_{32} = \frac{1+\nu}{E}\sigma_{23}; \quad \epsilon_{31} = \epsilon_{13} = \frac{1+\nu}{E}\sigma_{31}$$

## Solve these equations for a solution:

Strain Displacement: $\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$

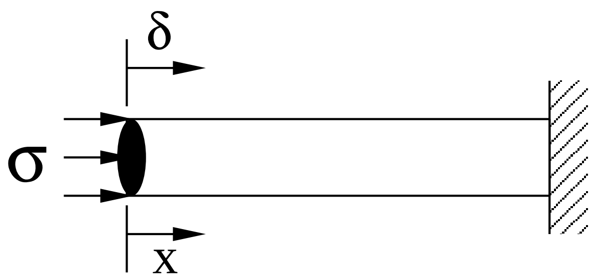
Equilibrium: $\frac{\partial \sigma_{ij}}{\partial x_j} = \rho a_i$

Constitutive Law: $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$

These are the *strong* forms of the governing equations $\rightarrow$ Differential equations that must be solved *pointwise* in time.

It is sometimes easier to solve these equations in an average sense than it is to solve them pointwise. This can be done with an integral statement of the governing equations. Then the integral over some volume of material satisfies the governing equations.

An energy method $\longrightarrow$ If we could find an equation for the energy in a body, then minimization of this equation would give us an equilibrium state.




$$u(x = L) = \delta$$

$$u(x = 0) = 0$$

$$\epsilon = \frac{\partial u}{\partial x} = \frac{\delta}{L}$$

$$\sigma = E\epsilon$$

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$$

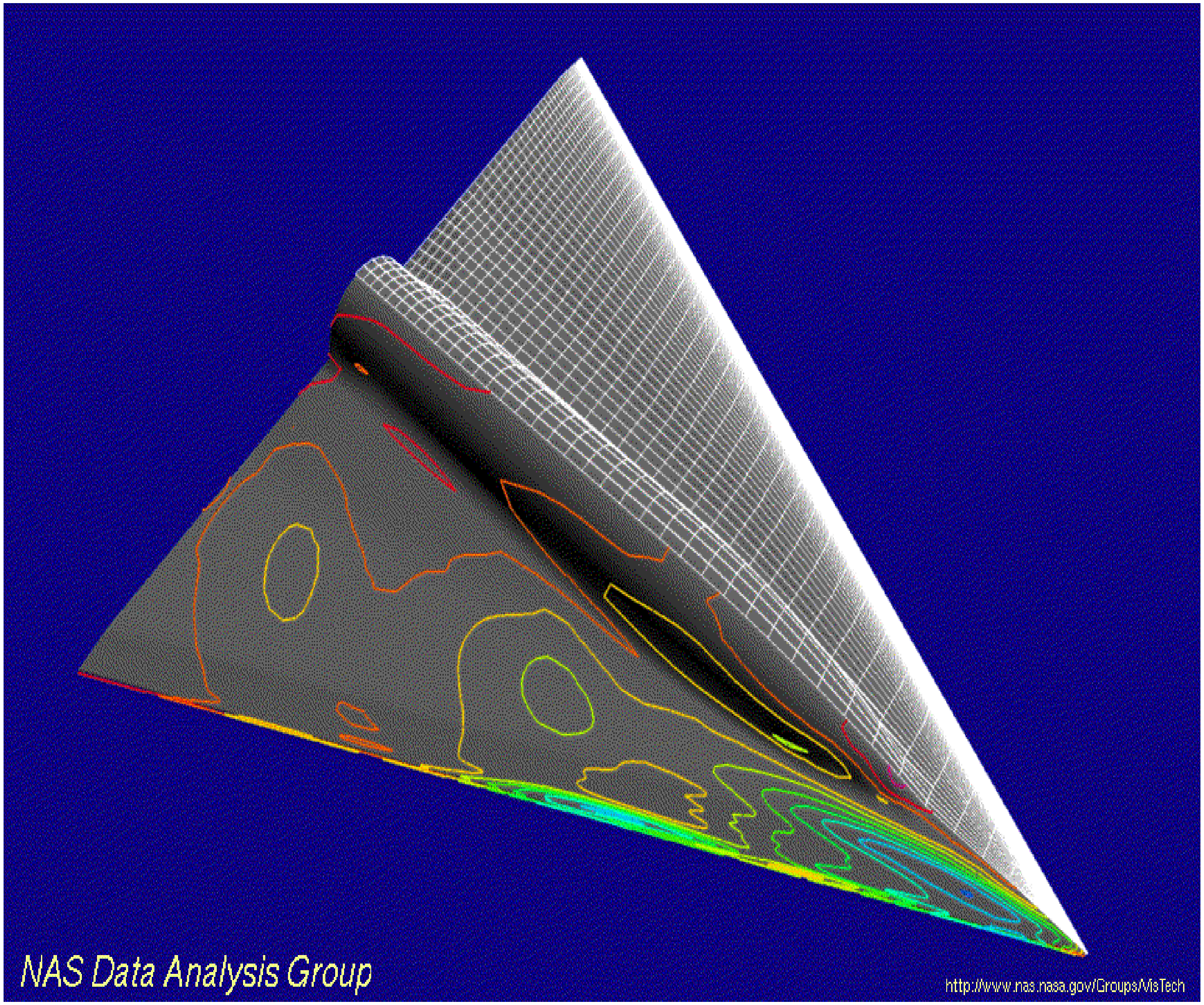
Lets perturb the solution by a small amount $\delta u$. $\delta u$ satisfies the boundary conditions but is otherwise arbitrary. These *virtual* displacements cause the internal and body forces to do *virtual* work that we'll call $\delta W$.

$$\delta W = \int_V \rho \frac{\partial^2 u}{\partial t^2} \delta u \, dV + \int_V \frac{\partial \sigma}{\partial x} \delta u \, dV$$

## Princple of Virtual Work

$$\delta W = \int_V \rho \frac{\partial^2 u}{\partial t^2} \delta u \, dV + \int_V \frac{\partial \sigma}{\partial x} \delta u \, dV$$

- PVW is a volumetric statement of the work done on a body by a set of virtual displacements.
- Equilibrium is a state of minimum energy, therefore, PVW must be a minimum for an equilibrium state.
  - In fact $\delta W = 0 \rightarrow \delta u$ cannot add energy to the system because there is only 1 equilibrium state in the vicinity of $u$.
- PVW is a function of a single variable $u$, methods of solution will yield $u$ as a result.
- In a similar manner, we could formulate an energy function with 2 or 3 basic variables
  - Two Field $\rightarrow u - p$; $u - \epsilon$
  - Three Field $\rightarrow u - p - \epsilon_V$; $u - \epsilon - \sigma$


NAS Data Analysis Group
http://www.nas.nasa.gov/Groups/VisTech

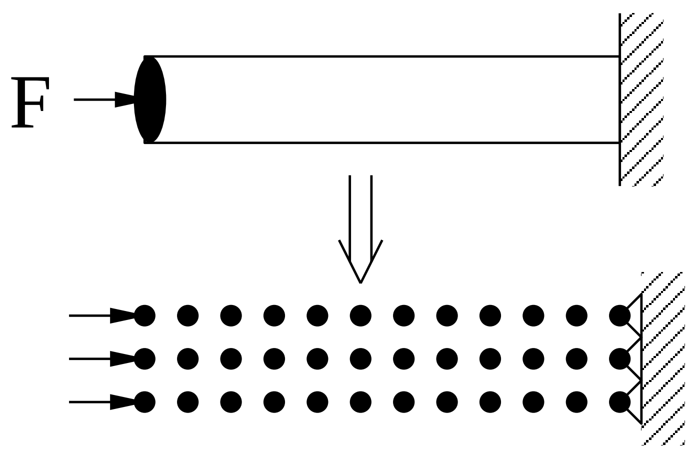
Numerical approximation to the strong form of the governing equations. Discretize the equations in space and time $\longrightarrow$ the equations are converted from pointwise differential equations to *finite difference* equations.

$$\frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \longrightarrow \frac{\Delta(\Delta u)}{(\Delta x)^2} = \rho \frac{\Delta(\Delta u)}{(\Delta t)^2}$$

FD has been a dominant method in numerical solution techniques. It is a natural approximation for the governing equations and the concept is simple.

In short, FD;

- Utilizes uniformly spaced grids of nodes,
- At the nodes, the necessary differences are approximated by the nodal value at that node and the adjacent nodes,
- This creates a system of algebraic equations,
- The system of equations is solved for the dependent variable.

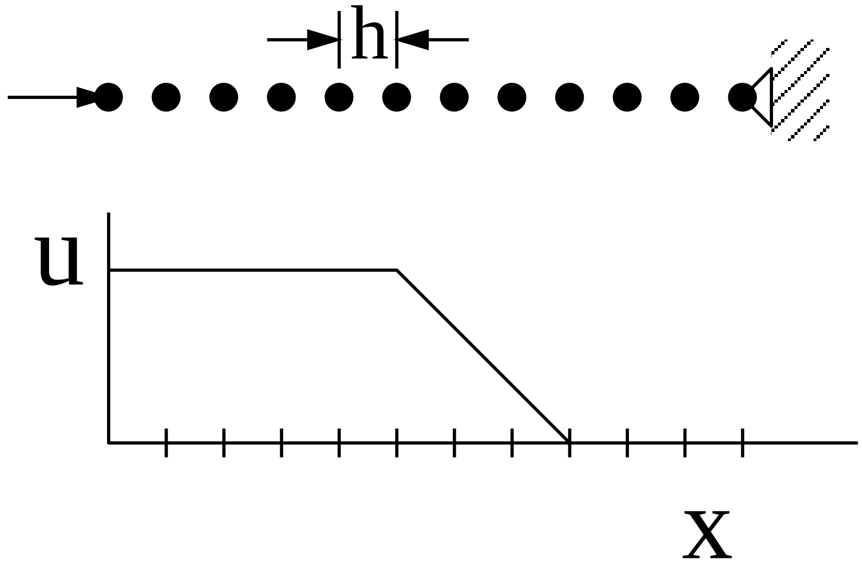



$$\frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \rightarrow \frac{\Delta(\Delta u)}{(\Delta x)^2} = \rho \frac{\Delta(\Delta u)}{(\Delta t)^2}$$

Taylor series approximation of $u(x+h)$:

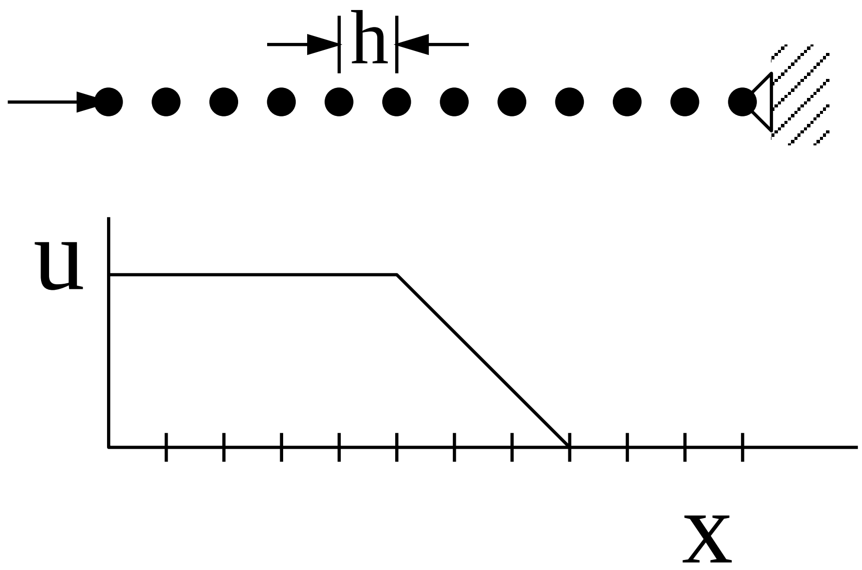
$$u(x+h) = u(x) + \frac{h}{1!}u'(x) + \frac{h^2}{2!}u''(x) + \ldots =$$

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} u^n(x) = u(x) + \frac{h}{1!}u'(x) + O(h^2)$$



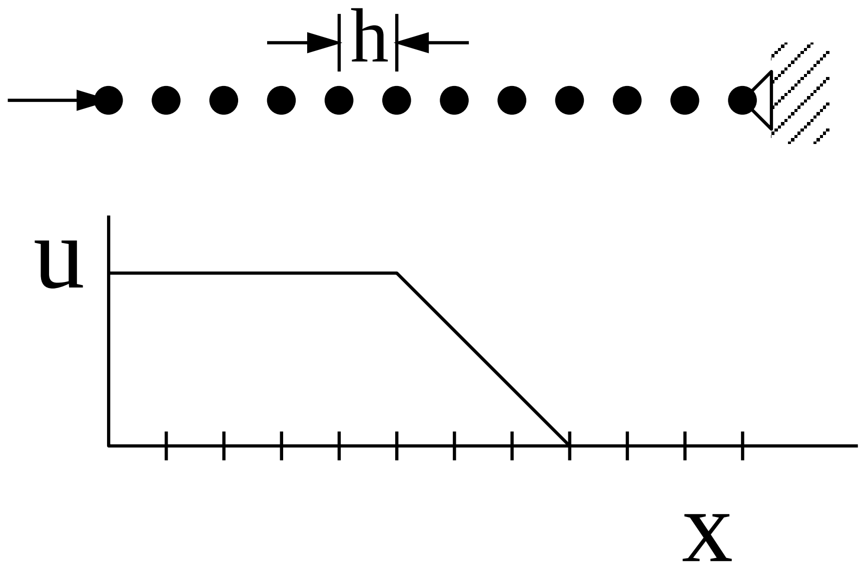

Forward difference approx. of $\frac{du}{dx}$:

$$\frac{du}{dx} \approx \frac{u(x+h) - u(x)}{h}$$




Backward difference use $u(x-h)$:

$$\frac{du}{dx} \approx \frac{u(x) - u(x-h)}{h}$$




Central difference:

$$\frac{du}{dx} \approx \frac{u(x+h) - u(x-h)}{2h}$$

Approximate $\frac{d^2u}{dx^2}$

$$u(x+h) \approx u(x) + \frac{h}{1!}u'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x)$$

$$u(x-h) \approx u(x) - \frac{h}{1!}u'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x)$$

$$u(x+h) + u(x-h) \approx 2u(x) + h^2u''(x)$$

$$\frac{d^2u}{dx^2} \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

In a similar manner, $\frac{du}{dt}$

Forward difference:

$$\frac{du}{dt} \approx \frac{u(x,t+l) - u(x,t)}{l}$$

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Backward difference:

$$\frac{du}{dt} \approx \frac{u(x,t) - u(x,t-l)}{l}$$

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Central difference:

$$\frac{du}{dt} \approx \frac{u(x,t+l) - u(x,t-l)}{2l}$$

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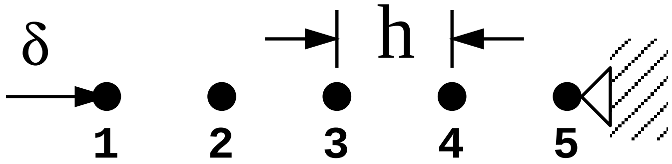
$$\frac{d^2u}{dt^2} \approx \frac{u(x,t+l) - 2u(x,t) + u(x,t-l)}{l^2}$$

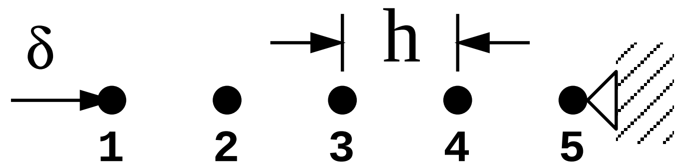
# Finite Difference - An Example




$$\frac{d^2u}{dx^2} = \frac{1}{h^2}\Big(u(x_{i-1}) - 2u(x_i) + u(x_{i+1})\Big) = 0$$

$$u(x_{i-1}) - 2u(x_i) + u(x_{i+1}) = 0$$

$$u(x_1) = \delta; \quad u(x_5) = 0$$




Node 2:

$$u(x_1) - 2u(x_2) + u(x_3) = 0$$

$$-2u(x_2) + u(x_3) = -\delta$$

Node 3:

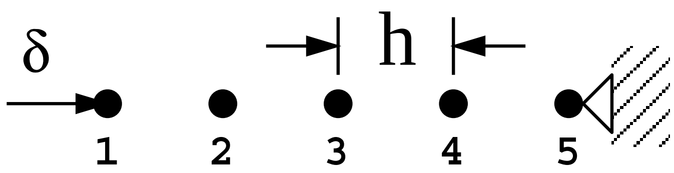
$$u(x_2) - 2u(x_3) + u(x_4) = 0$$

Node 4:

$$u(x_3) - 2u(x_4) + u(x_5) = 0$$

$$u(x_3) - 2u(x_4) = 0$$

# Finite Difference - An Example




$$-2u(x_2) + u(x_3) = -\delta$$

$$u(x_2) - 2u(x_3) + u(x_4) = 0$$

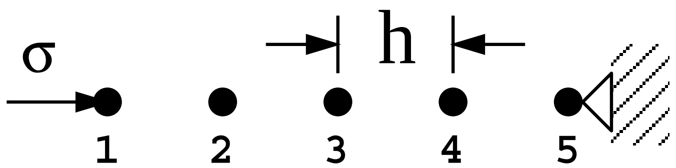
$$u(x_3) - 2u(x_4) = 0$$

---

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix} = \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -\delta \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix} = \begin{bmatrix} \frac{3}{4}\delta \\ \frac{1}{2}\delta \\ \frac{1}{4}\delta \end{bmatrix}$$



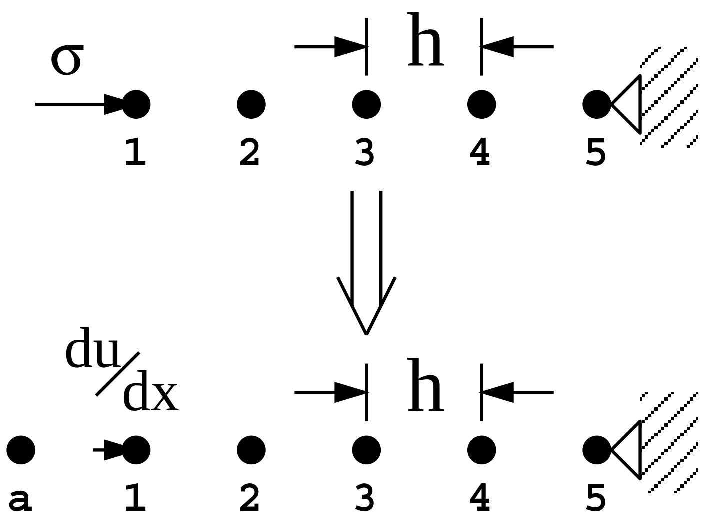

$$\sigma(x_1) = \sigma$$

$$E\frac{du}{dx} = \sigma$$

$$\frac{du}{dx} = \frac{\sigma}{E}$$

-Neumann (natural) BCs are naturally applied $\rightarrow$ they specify the solution variable.

-Dirichlet (essential) BCs are a little bit more difficult $\rightarrow$ they specify a derivative of the solution variable.




-Add node *a* to the mesh.

At node 1:

$$u(x_a) - 2u(x_1) + u(x_2) = 0$$

The central difference for $\frac{du}{dx}$

$$\frac{du}{dx} = \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$$

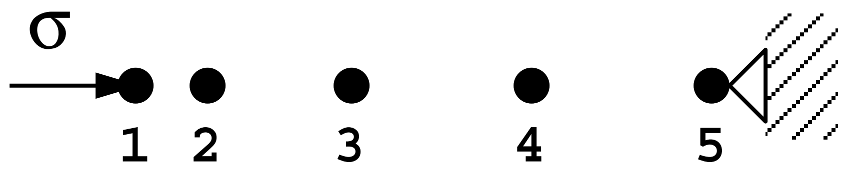
$$\frac{u(x_2) - u(x_a)}{2h} = \frac{\sigma}{E}$$

$$u(x_a) = u(x_2) - 2h\frac{\sigma}{E}$$

---

$$u(x_a) - 2u(x_1) + u(x_2) = 0 \rightarrow -2u(x_1) + 2u(x_2) = 2h\frac{\sigma}{E}$$

## Irregular spacing




$$\left(\frac{du}{dx}\right)_{1-2} = \frac{u(x_2) - u(x_1)}{h_{1-2}}$$

$$\left(\frac{d^2u}{dx^2}\right)_2 = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{\frac{u(x_3)-u(x_2)}{h_{2-3}} - \frac{u(x_2)-u(x_1)}{h_{1-2}}}{h_{1-2} + h_{2-3}}$$

## Time Dependence - The Wave Equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Explicit Method:

$$\frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{h^2} =$$

$$\frac{1}{c^2}\frac{u(x_i,t+\Delta t) - 2u(x_i,t) + u(x_i,t-\Delta t)}{\Delta t^2}$$

$$\frac{c^2}{\Delta t^2 h^2}\left[u(x_{i+1},t) + u(x_i,t) + u(x_{i-1},t)\right]$$

$$+2u(x_i,t) - u(x_i,t-\Delta t) = u(x_i,t+\Delta t)$$

## Time Dependence - The Wave Equation

### Implicit Method:

$$\frac{u(x_{i+1}, t+\Delta t) - 2u(x_i, t+\Delta t) + u(x_{i-1}, t+\Delta t)}{h^2} =$$

$$\frac{1}{c^2}\frac{u(x_i, t+\Delta t) - 2u(x_i, t) + u(x_i, t-\Delta t)}{\Delta t^2}$$

-The result of the above approximation is a system of equations that can be solved for $u(x_i, t+\Delta t)$.
-Explicit schemes formulate a solution at a node in terms of *explicitly* known quantities.
-Implicit schemes formulate a solution at a node in terms of quantities that are *implied*.
-Explicit schemes are generally limited to very small $\Delta t$ whereas, implicit schemes are not.

Assume a general solution (*e.g.* $u^*$ to the governing equation. The difference between the expected solution and the assumed solution is the *residual*.

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u^*}{\partial x^2} = R$$

Weight $R$ to zero in some manner so that the error between the assumed solution and the actual solution is small over the body of interest.

The governing equation:

$$\frac{\partial^2 u}{\partial x^2} = 0$$

The residual:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u^*}{\partial x^2} = \frac{\partial^2 u^*}{\partial x^2} = R$$

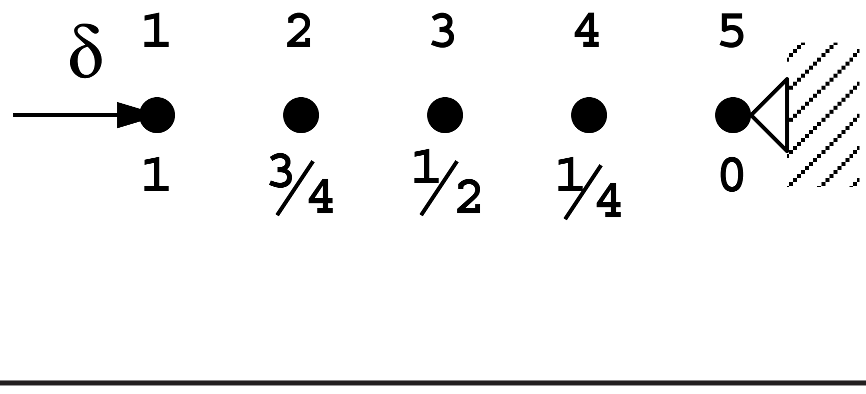
The trick is to try to force:

$$R = 0 \quad \forall \quad x \in [\Omega]$$

-If this is true, $u^*$ is an exact solution.

-This is rarely the case so we want to get close.

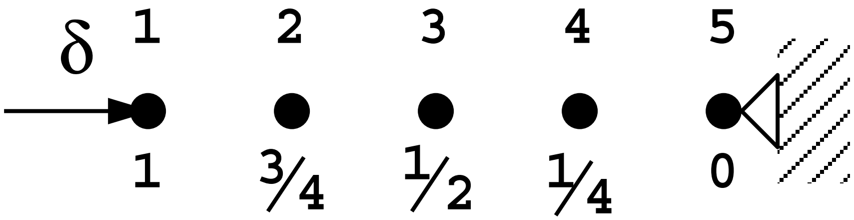
-In general there are two ways to approximate $R = 0$ in the domain; (1)At a specific number of points, (2)In a specific number of sub-volumes.




$$u^* = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3$$

$$R = \frac{d^2 u^*}{dx^2} = 2\, a_2 + 6\, a_3\, x$$

# Solution Methods - Collocation




$$u^* = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3$$

$$R = \frac{d^2u^*}{dx^2} = 2\,a_2 + 6\,a_3\,x$$

---

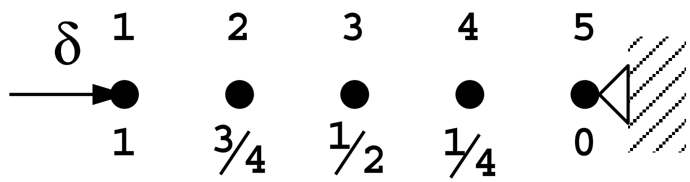
$$R_2 = 2\,a_2 + 6\,a_3\left(\frac{3}{4}\right)$$

$$R_3 = 2\,a_2 + 6\,a_3\left(\frac{1}{2}\right)$$

$$R_4 = 2\,a_2 + 6\,a_3\left(\frac{1}{4}\right)$$

$$u^*(1) = \delta = a_0 + a_1 + a_2 + a_3$$

$$u^*(0) = 0 = a_0$$




$$2\,a_2 + 6\,a_3\left(\frac{3}{4}\right) = 0$$

$$2\,a_2 + 6\,a_3\left(\frac{1}{2}\right) = 0$$

$$2\,a_2 + 6\,a_3\left(\frac{1}{4}\right) = 0$$

$$a_0 + a_1 + a_2 + a_3 = \delta$$

$$a_0 = 0$$

-5 equations , 4 unknowns $\rightarrow$ over-contrained.

-Increase order of $u^*$.

-Remove nodes from the domain.

-Find the *best* solution (*e.g. least squares*)

For simplicity, remove node 2

$$2\,a_2 + 3\,a_3 = 0$$

$$2\,a_2 + \frac{3}{2}a_3 = 0 \qquad \longrightarrow \qquad \begin{bmatrix} 0 & 2 & 3 \\ 0 & 2 & \frac{3}{2} \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \delta \end{Bmatrix}$$

$$a_1 + a_2 + a_3 = \delta$$

---

$$\left[C^{-1}\right]\{R\} = \{a\} \rightarrow \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & 1 \\ -\frac{1}{2} & 1 & 0 \\ \frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \delta \end{Bmatrix} = \begin{Bmatrix} \delta \\ 0 \\ 0 \end{Bmatrix}$$

$$u^* = \delta x$$

Weighted Residual methods have the general form:

$$\int_{x_i} W(x)\, R(x)\, dx = 0$$

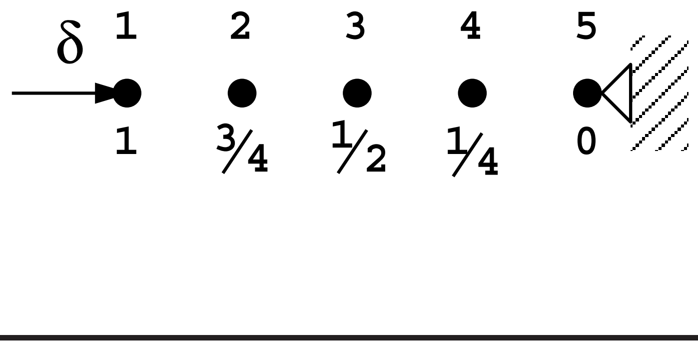
Collocation *can* be viewed in the above form with

$$W(x) = \delta(x_i) = \begin{cases} 1; & x = x_i \\ 0; & x \neq x_i \end{cases}$$

In the subdomain method, the problem is divided into many subdomains and the average residual in each subdomain is forced to be zero.

$$W(x) = \frac{1}{\Delta x_i}$$

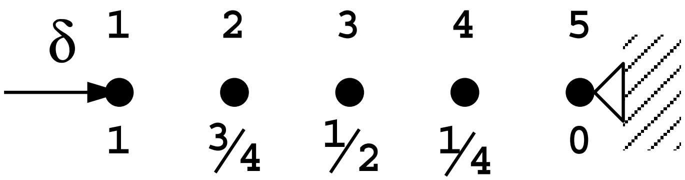
$$\frac{1}{\Delta x_i} \int_{x_i} R(x)\, dx = 0$$




$$u^* = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$R = \frac{d^2 u^*}{dx^2} = 2\,a_2 + 6\,a_3\,x$$

$$\int R dx = 2\,a_2\,x + 3\,a_3\,x^2$$




$$\int R dx = 2\,a_2\,x + 3\,a_3\,x^2$$

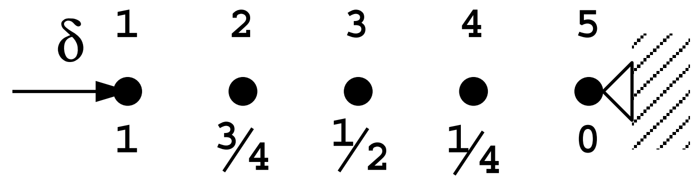
$$D_{1-2}: \quad \left(2\,a_2\,x + 3\,a_3\,x^2\right)_{3/4}^{1} = 2\,a_2 \;+ \frac{21}{4}\,a_3 = 0$$

$$D_{2-3}: \quad \left(2\,a_2\,x + 3\,a_3\,x^2\right)_{1/2}^{3/4} = 2\,a_2 \;+ \frac{15}{4}\,a_3 = 0$$

$$D_{3-4}: \quad \left(2\,a_2\,x + 3\,a_3\,x^2\right)_{1/4}^{1/2} = 2\,a_2 \;+ \frac{9}{4}\,a_3 = 0$$

$$D_{4-5}: \quad \left(2\,a_2\,x + 3\,a_3\,x^2\right)_{0}^{1/4} = 2\,a_2 \;+ \frac{3}{4}\,a_3 = 0$$

# Solution Methods - Subdomain




$$2\,a_2 \;+ \frac{21}{4} a_3 = 0$$

$$2\,a_2 \;+ \frac{15}{4} a_3 = 0$$

$$2\,a_2 \;+ \frac{9}{4} a_3 = 0$$

$$2\,a_2 \;+ \frac{3}{4} a_3 = 0$$

$$a_0 + a_1 + a_2 + a_3 = \delta$$

$$a_0 = 0$$

-6 equations , 4 unknowns.
-Reduce order of guess to
$u^* = a_0 + a_1\,x$.

Then,

$$a_1 = \delta$$

-We have been using $u^* = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ to approximate the solution throughout the body.

-Another method is to approximate the solution in each domain:

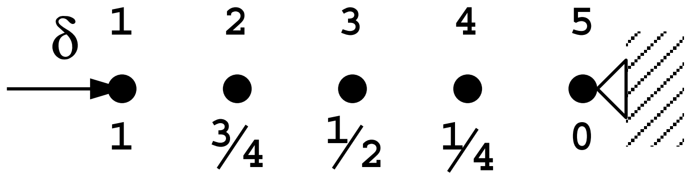
$$u^* = \sum_{i=1}^{n} N_i a_i$$

Where $n$ is the number of nodes in each domains, $N_i$ are interpolation functions, and $a_i$ are unknown node parameters (*e.g.* nodal displacements).

-With this method, the dependent variable is determined at each node and interpolated between nodes with the functions $N_i$.

-The solution is thus approximated in each domain and not throughout the body.

-We could have used this method for the collocation and subdomain methods, and avoided the overconstraint problems.

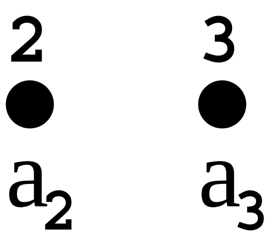



For the local coordinate system
$x_2 = 0$, $x_3 = \Delta x$:

$$u_{x_2}^{x_3} = \left(1 - \frac{x}{\Delta x}\right) a_2 + \left(\frac{x}{\Delta x}\right) a_3$$

$$\sum_{i=1}^{2} N_i a_i = \left[\begin{array}{cc} 1 - \frac{x}{\Delta x} & \frac{x}{\Delta x} \end{array}\right] \left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]$$

Then for our 2 node element,




$$N_1 = 1 - \frac{x}{\Delta x}; \quad N_2 = \frac{x}{\Delta x}$$

Note the properties of $N_i$

$$N_1(x_1) = 1; \quad N_1(x_2) = 0$$

$$N_2(x_1) = 0; \quad N_2(x_2) = 1$$

In Galerkin's method, the weights on the residual are the interpolation functions $N_i$

$$\int_{x_i} W(x)\, R(x)\, dx = 0$$

$$R(x) = \frac{d^2 u^*}{dx^2} = \frac{d^2}{dx^2} \sum N_i a_i$$

$$\int_{x_i} W(x)\, R(x)\, dx = \int_{x_i} N_j(x)\, \frac{d^2}{dx^2} \sum N_i a_i\, dx$$

$N_i$ is linear in $x$, hence the second derivative vanishes. Through integration by parts:

$$\int_{x_i} N_j \frac{d^2}{dx^2} \sum N_i a_i \, dx = \int_{x_i} \frac{d}{dx} N_i \frac{d}{dx} \sum N_i a_i \, dx = 0$$

Aside: $\int u \, dv = uv - \int v \, du$

$$dv = \frac{d^2}{dx^2} \sum N_i a_i \quad u = N_i$$

$$v = \frac{d}{dx} \sum N_i a_i \quad du = \frac{d}{dx} N_i$$

And $u \cdot v$ vanishes by virtue of the properties of $N_i$

We can rewrite the following equation in a shortened notation

$$\int_{x_i} \frac{d}{dx} N_i \frac{d}{dx} \sum N_i \, a_i \, dx = 0$$

Using the following definitions

$$[K] = \int_{x_i} \frac{d}{dx} N_i \frac{d}{dx} \sum N_i dx = \int_{x_i} \begin{bmatrix} \frac{1}{\Delta x} & -\frac{1}{\Delta x} \\ -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{bmatrix} dx$$

$$\{u\} = a_i = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

Then,

$$[K]\{u\} = \{0\}$$

Element 1 $\rightarrow$ nodes 1,2:

$$\int_{x_1} \begin{bmatrix} \frac{1}{\Delta x_1} & -\frac{1}{\Delta x_1} \\ -\frac{1}{\Delta x_1} & \frac{1}{\Delta x_1} \end{bmatrix} dx \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Element 2 $\rightarrow$ nodes 2,3:

$$\int_{x_2} \begin{bmatrix} \frac{1}{\Delta x_2} & -\frac{1}{\Delta x_2} \\ -\frac{1}{\Delta x_2} & \frac{1}{\Delta x_2} \end{bmatrix} dx \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

And so on ...

The global system of equations is then

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1+1 & -1 & 0 & 0 \\ 0 & -1 & 1+1 & -1 & 0 \\ 0 & 0 & -1 & 1+1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Boundary conditions: $a_1 = \delta$, $a_5 = 0 \rightarrow$ remove rows 1 and 5, then rows 2 and 4 become:

$$2\,a_2 - a_3 = \delta; \quad -a_3 + 2\,a_4 = 0$$

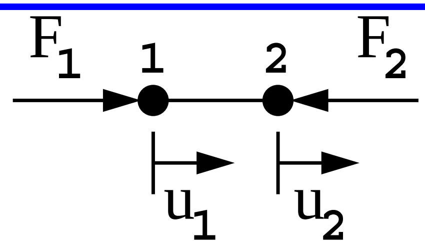
The matrix equation is then

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} \delta \\ 0 \\ 0 \end{Bmatrix}$$

-The basic philosophy of the finite element method (FEM) is to divide the body into (*somewhat*) arbitrary and convenient subdomains (*i.e.* finite elements).
-The weak form of the governing equations is then enforced over each of these elements.
-The *global* solution is determined by assembling the numerical approximation of each element into a global system of linear equations and then solving the resulting matrix equation.

$$[K]\{u\} = \{F\}$$

-The Galerkin method is one way to derive the FEM equations.
-The Direct Method formulates equations based on $F(\delta)$.
-The variational approach manipulates the weak form of the governing equations.

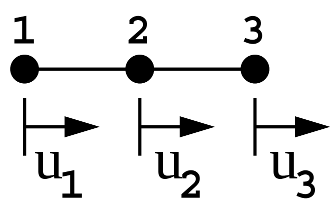



$$\delta = \frac{FL}{AE} \quad \rightarrow \quad F = \frac{AE}{L}\delta$$

$$F_1 = \frac{A_1E_1}{L_1}\delta_{1-2} = \frac{A_1E_1}{L_1}(u_1 - u_2)$$

$$F_2 = \frac{A_1E_1}{L_1}\delta_{2-1} = \frac{A_1E_1}{L_1}(-u_1 + u_2)$$

$$\frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$




Element 2:

$$\frac{A_2E_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

-The equations are assembled in the usual manner (this is left to the interested student).

Look at the case for $\{F\} = 0$. Then for element 1,

$$\frac{1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\left\{\begin{array}{c} u_1 \\ u_2 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \end{array}\right\}$$

-This is the same equation as the Galerkin method.

$$\boxed{\text{The coolest thing since sliced bread.}}$$

Recall the principle of virtual work;

$$\int_x \frac{d\sigma}{dx}\,\delta u\,dx = \delta W = 0$$

Look at

$$\frac{d}{dx}(\sigma\delta u) = \frac{d\sigma}{dx}\delta u + \frac{d\delta u}{dx}\sigma$$

$$\frac{d\sigma}{dx}\delta u = \frac{d}{dx}(\sigma\delta u) - \frac{d\delta u}{dx}\sigma$$

Then PVW becomes

$$\int_x \frac{d}{dx}(\sigma\delta u) - \frac{d\delta u}{dx}\sigma\,dx = 0$$

Look at;

$$\frac{d\delta u}{dx} = \delta \frac{du}{dx} = \delta \epsilon$$

PVW becomes

$$\int_x \frac{d}{dx}(\sigma \delta u) - \delta\epsilon\, \sigma\, dx = 0$$

Recall the divergence theorem;

$$\int_V \frac{d}{dx} F\, dV = \int_S F \cdot n\, dS$$

PVW becomes

$$\int_s \sigma \cdot n \delta u ds - \int_x \delta\epsilon\, \sigma\, dx = 0$$

Recall our Galerkin displacement interpolation;

$$u^* = \sum_{i=1}^{n} N_i a_i$$

Then;

$$\delta u = \sum N_i \, \delta a_i$$

$$\epsilon = \sum \frac{dN_i}{dx} a_i \quad \rightarrow \quad \delta\epsilon = \sum \frac{dN_i}{dx} \delta a_i$$

Also recall the constitutive relation;

$$\sigma = E\epsilon \quad \rightarrow \quad \sigma \approx E \sum \frac{dN_j}{dx} a_j$$

PVW becomes

$$\int_s \sigma \cdot n \sum N_i \delta a_i ds = \int_x \sum \frac{dN_i}{dx} \delta a_i \, E \sum \frac{dN_j}{dx} a_j \, dx$$

PVW now

$$\int_s \sum \left[\sigma \cdot n N_i\right] \delta a_i ds = \int_x \sum \left[\frac{dN_i}{dx} E \sum \frac{dN_j}{dx} a_j\right] \delta a_i \, dx$$

-In the development of the PVW we developed a statement of the work for a small *arbitrary* variation ($\delta u$) in the actual solution.
-Since the variation in the actual solution is arbitrary, then the variation in the approximation ($\delta a$)is arbitrary.
-We can then make $\delta a$ anything,lets make it independent of space and make $\delta a_{LHS} = \delta a_{RHS}$

$$\int_s \sigma \cdot n \, N_i \, ds = \int_x \frac{dN_i}{dx} E \sum \frac{dN_j}{dx} a_j \, dx$$

PVW now

$$\int_s \sigma \cdot n \, N_i \, ds = \int_x \frac{dN_i}{dx} E \sum \frac{dN_j}{dx} a_j \, dx$$

Rewriting the above in shortened matrix notation

$$[K] \{u\} = \{F\}$$

Where

$$[K] = \int_x \frac{dN_i}{dx} E \frac{dN_j}{dx} \, dx$$

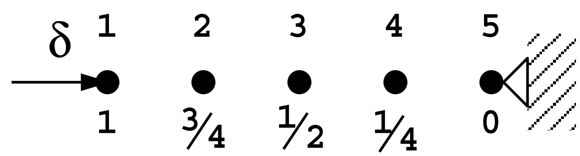
$$\{u\} = a_j$$

$$\{F\} = \int_s \sigma \cdot n \, N_i \, ds$$

-We have thus derived the standard finite element equations,

$$[K]\{u\} = \{F\}$$

which, if we took our 5 node 4 element example problem:




would look amazingly similar to the finite difference and Galerkin matrix equations.

-While this coincidence is only true for special cases, the methods are intimately connected in that they provide an approximation to the governing equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

# What about Dynamics?

-We could have used a similar procedure to derive a matrix equation of the form,

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F\}$$

where,

$$[K] = \int_x \frac{dN_i}{dx} E \frac{dN_j}{dx}\, dx$$

$$[M] = \int_x \rho \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx$$

$$[C] = \int_x \kappa \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx$$

> This equation looks very similar to the generalized MDOF equation except now we have a representation of the continuous mass, damping and stiffness.

# How do we solve these equations?

-In our previous examples we formed $[K]\{u\} = \{F\}$.
-It is a simple concept to find $\{u\} = [K]^{-1}\{F\}$.
-This is often effective and efficient given the symmetric and sparse nature of $[K]$.

-What about $[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F\}$?
-Similar to spacial discretization, let's try to find solutions at specific times.
-Two principle methods $\rightarrow$ explicit and implicit integration.

# Explicit integration in brief

Central difference approximation to $\ddot{u}$ and $\dot{u}$

$$\ddot{u} \approx \frac{1}{\Delta t^2}(u(t-\Delta t) - 2u(t) + u(t+\Delta t))$$

$$\dot{u} \approx \frac{1}{2\Delta t}(u(t+\Delta t) - u(t-\Delta t))$$

Substituting in to $[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F\}$:

$$\left[\frac{1}{\Delta t^2}[M] + \frac{1}{2\Delta t}[C]\right]\{U(t+\Delta t)\} =$$

$$\{F(t)\} - \left[[K] - \frac{2}{\Delta t^2}[M]\right]\{U(t)\} - \left[\frac{1}{\Delta t^2}[M] - \frac{1}{2\Delta t}[C]\right]\{U(t-\Delta t)\}$$

This is a matrix equation that can be solved for $\{U(t+\Delta t)\}$. This method is conditionally stable $\rightarrow$ the stable time step, $\Delta t$, must be less than the time it takes a wave to traverse one element. For steel, the wave speed 200,000in/s. For a 1 in. element $\Delta t < 5 \times 10^{-6}$ s.

$$\left[\frac{1}{\Delta t^2}[M] + \frac{1}{2\Delta t}[C]\right]\{U(t+\Delta t)\} =$$

$$\{F(t)\} - \left[[K] - \frac{2}{\Delta t^2}[M]\right]\{U(t)\} - \left[\frac{1}{\Delta t^2}[M] - \frac{1}{2\Delta t}[C]\right]\{U(t-\Delta t)\}$$

-Lump the mass matrix.

-Ignore damping.

-Note that $[K]\{U\} = \{F\}$.

$$\left[\frac{1}{\Delta t^2}[M]\right]\{U(t+\Delta t)\} = \{R\}$$

Where $\{R\}$ is a vector of all forces acting on a node.

This looks very much like $\sum F = ma$ applied to each node

# Implicit integration - Newmark Beta

-Assuming $[M]$, $[C]$, and $[K]$ are constant over $\Delta t$, an incremental form of the governing equation can be written as:

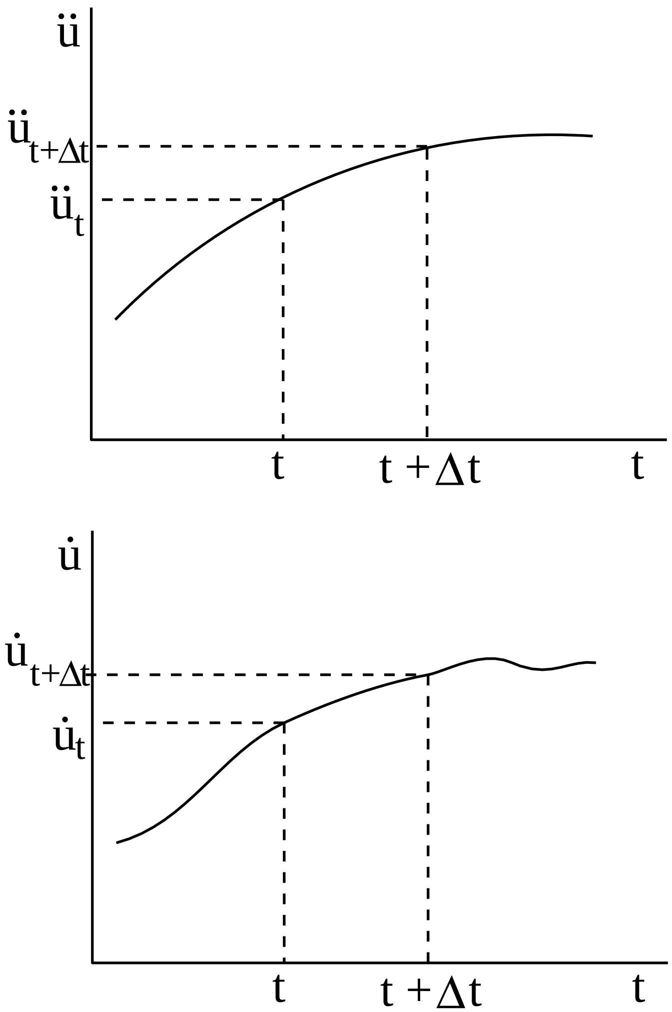
$$[M]\left\{\ddot{u}_{t+\Delta t}-\ddot{u}_t\right\}+[C]\left\{\dot{u}_{t+\Delta t}-\dot{u}_t\right\}+[K]\left\{u_{t+\Delta t}-u_t\right\}=\left\{F_{t+\Delta t}-F_t\right\}$$

-Grouping things we don't know on the LHS and things we do know on the RHS:

$$[M]\left\{\ddot{u}_{t+\Delta t}\right\}+[C]\left\{\dot{u}_{t+\Delta t}\right\}+[K]\{\Delta u\}=[M]\{\ddot{u}_t\}+[C]\{\dot{u}_t\}+\{\Delta F\}$$

-We would like to formulate the problem in terms of a single unknown $\Delta u$. Let's see what we can do with the $\ddot{u}-\dot{u}-u$ relationship and see if we can get $\ddot{u}_{t+\Delta t}$ and $\dot{u}_{t+\Delta t}$ in terms of $\Delta u$ and known values.

# Implicit integration - Newmark Beta




Using the trapezoidal rule:

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \frac{1}{2}\left(\ddot{u}_{t+\Delta t} + \ddot{u}_t\right)\Delta t$$

---

$$u_{t+\Delta t} = u_{t+\Delta t} + \frac{1}{2}\left(\dot{u}_{t+\Delta t} + \dot{u}_t\right)\Delta t$$

$$\Downarrow$$

$$\Delta u = \frac{1}{2}\left(\dot{u}_{t+\Delta t} + \dot{u}_t\right)\Delta t$$

$$\Delta u = \frac{1}{2}\left(\dot{u}_{t+\Delta t} + \dot{u}_t\right)\Delta t \quad \rightarrow \quad \dot{u}_{t+\Delta t} = \frac{2}{\Delta t}\Delta u - \dot{u}_t$$

---

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \frac{1}{2}\left(\ddot{u}_{t+\Delta t} + \ddot{u}_t\right)\Delta t = \frac{2}{\Delta t}\Delta u - \dot{u}_t$$

$$\Downarrow$$

$$\ddot{u}_{t+\Delta t} = \frac{4}{\Delta t^2}\Delta u - \frac{4}{\Delta t}\dot{u}_t - \ddot{u}_t$$

## Implicit integration - Newmark Beta

$$[M]\left\{\frac{4}{\Delta t^2}\Delta u-\frac{4}{\Delta t}\dot{u}_t-\ddot{u}_t\right\}+[C]\left\{\frac{2}{\Delta t}\Delta u-\dot{u}_t\right\}+[K]\{\Delta u\}=$$

$$[M]\{\ddot{u}_t\}+[C]\{\dot{u}_t\}+\{\Delta F\}$$

$$\Downarrow$$

$$\left[\frac{4}{\Delta t^2}[M]+\frac{2}{\Delta t}[C]+[K]\right]\{\Delta u\}=$$

$$2[M]\{\ddot{u}_t\}+\left[\frac{4}{\Delta t}[M]+2[C]\right]\{\dot{u}_t\}+\{\Delta F\}$$

$$\Downarrow$$

$$[K]\{\Delta u\}=\{\mathcal{F}\}$$

- Boundary integral - often called boundary element method (BEM).

- The development starts with the weak form.

- Recall we used the divergence theorem to transform part of PVM from a volume integral to a surface integral.

- We can do something similar to transform PVW completely into a surface integral.

- Then the EOM for the entire body is solved by an integration performed on the surface of the body.

- This can significantly reduce the size of the matrix problem.

- One drawback is the matrix is fully dense.

## Other methods - Meshless methods

- Meshless methods *can* be thought of as the finite element method.

- The difference is that the nodes that define an element are found during the simulation.

- In other words, the are meshless (or element free), in the sense that the user doesn't have to define the element.

- One of the primary benefits of this type of formulation is that mesh evolution is natural.

- One drawback is that they are slower than molasses in January.

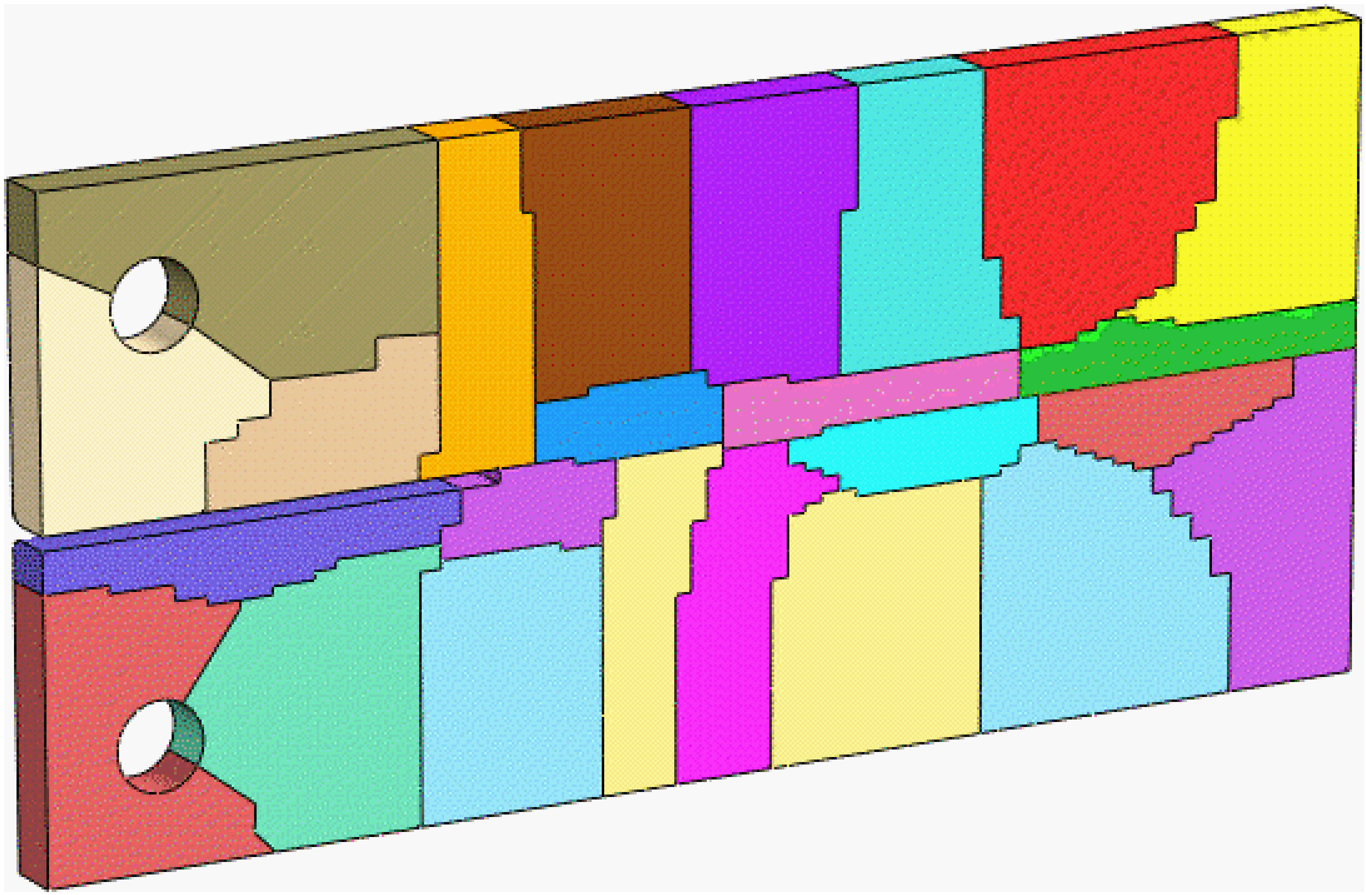
- This is a current *hot* research topic and advances could make them competitive with finite elements.

$\boxed{\text{This all sounds so simple - then why is it so hard?}}$

- The major problem in numerical simulation of anything is the question of how well does the model represent the reality.

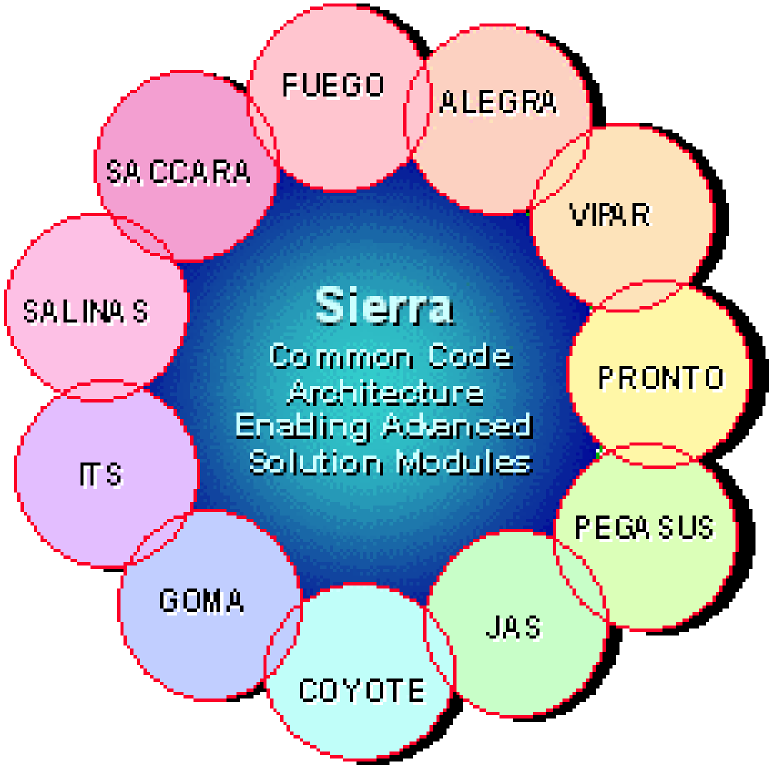
# Current Research - Speed by Parallelization

# Current Research - Parallelization

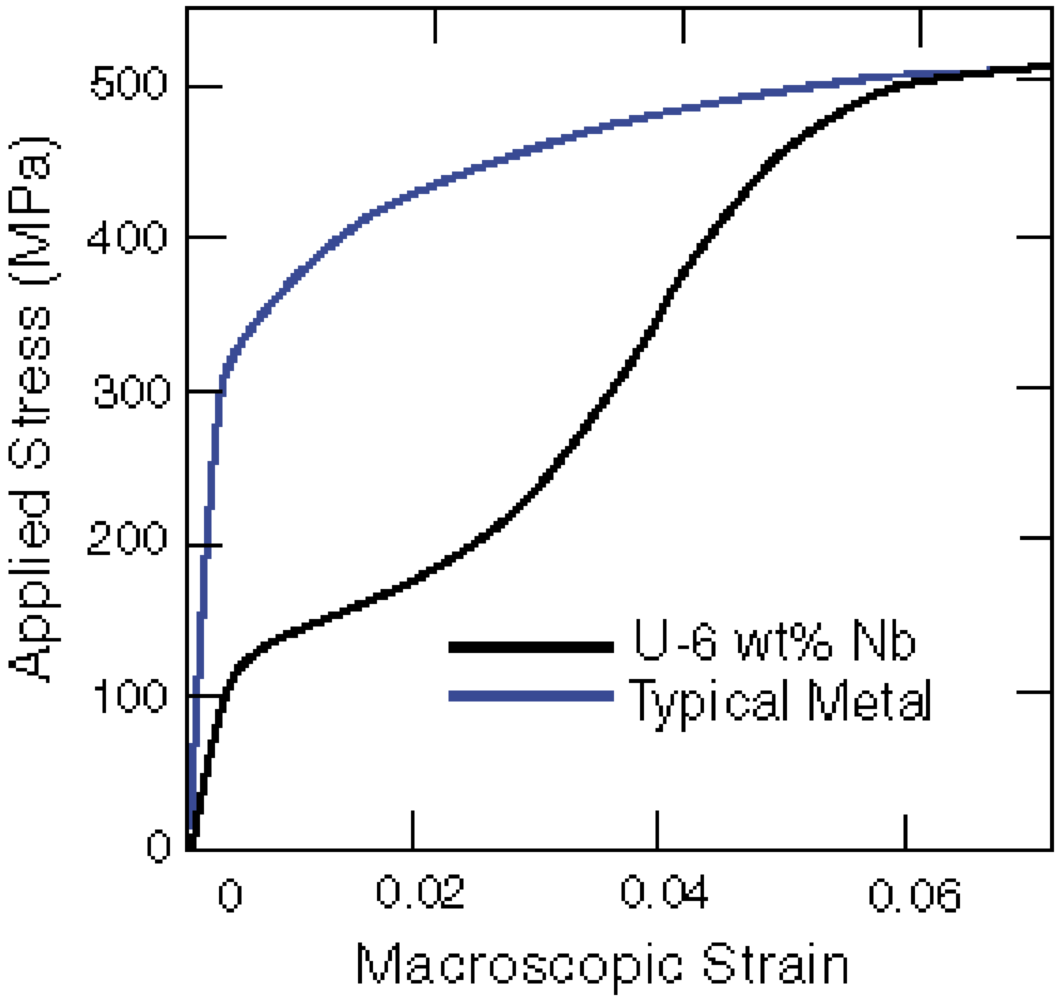


# Current Research - Multi-Scale Physics/Code Coupling

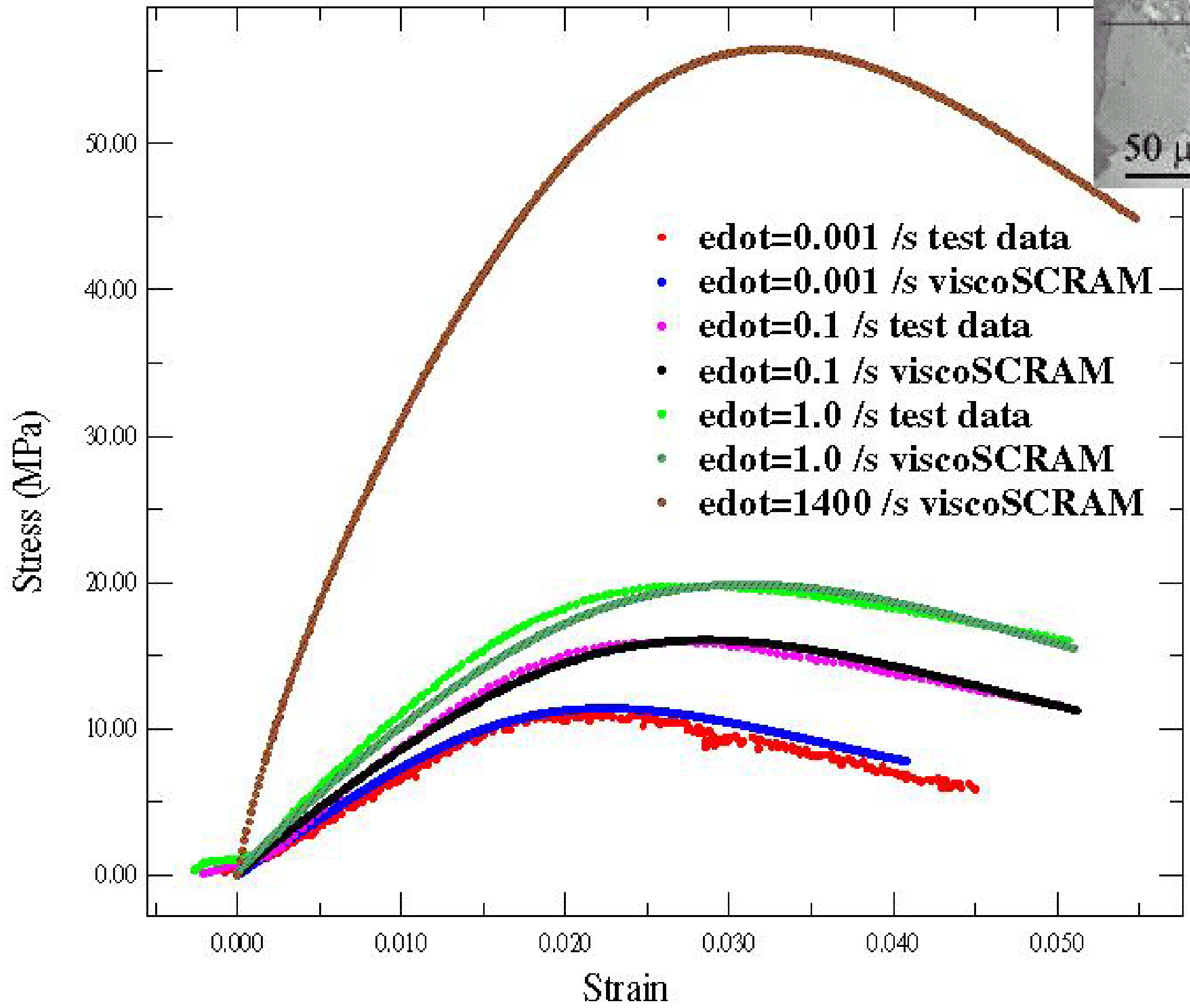
- VIPAR
  - •Parachute performance code, vortex method with transient dynamics
- PRONTO
  - •Transient dynamics
  - •Lagrangian solid mechanics
- JAS
  - •Quasistatic solid mechanics
- COYOTE
  - •Thermal mechanics with chemistry
- GOMA
  - •Incompressible fluid mechanics with free surfaces
- PEGASUS
  - •Neutron Tube Physics
- FUEGO
  - •Fire simulation
- SALINAS
  - •Linear structural dynamics
- SACCARA
  - •Compressible fluid mechanics
- ITS
  - •Radiation transport

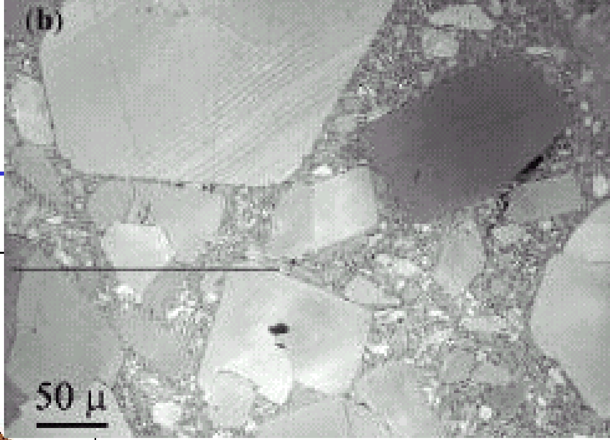


# Current Research - Material Models



# Current Research - Material Models



# Material Modeling

$$\dot{\varepsilon}_{ij} = \frac{1}{3}\dot{\varepsilon}_{kk}\delta_{ij} + \dot{e}_{ij}$$

Strain rate decomposition

$$\dot{e}_{ij} = \dot{e}_{ij}^{v\varepsilon} + \dot{e}_{ij}^{c}$$

Deviatoric strain rate decomposition

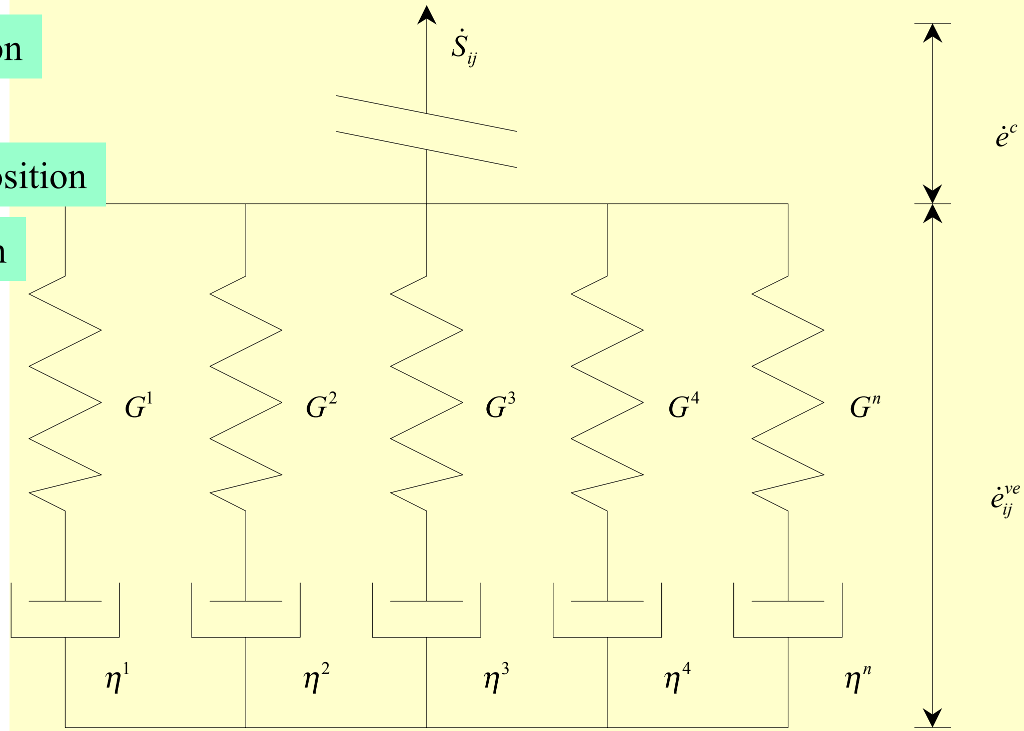
$$\dot{\sigma}_{ij} = \dot{S}_{ij} + \dot{\sigma}_{m}\delta_{ij}$$

Stress rate decomposition

$$\dot{\sigma}_{m} = K\dot{\varepsilon}_{ii}$$

$$S = \sum_{n=1}^{N} S^{(n)}$$

$$\tau = \frac{\eta}{G}$$

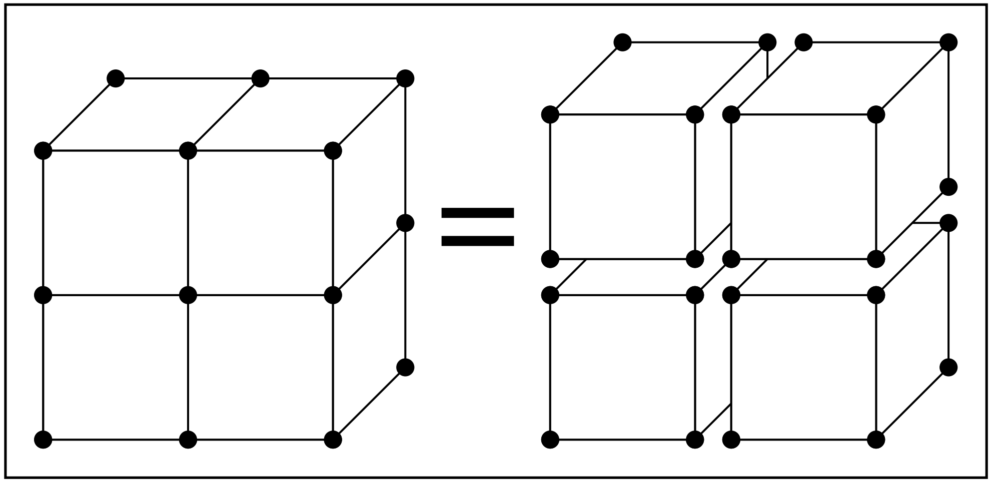


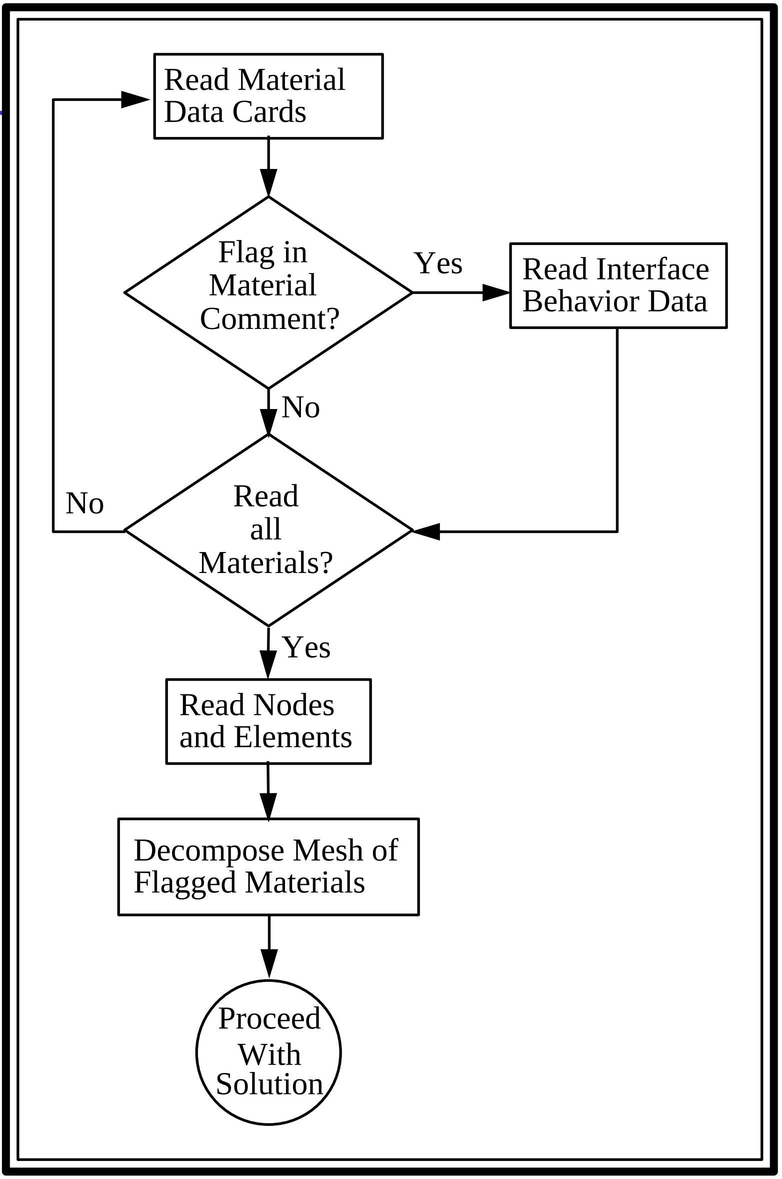

$$\dot{S}_{ij} = \frac{2G\dot{e}_{ij} - \sum_{1}^{N}\frac{S_{ij}^{(n)}}{\tau^{(n)}} - 3\left(\frac{c}{a}\right)^{2}\frac{\dot{c}}{a}S_{ij}}{1+\left(\frac{c}{a}\right)^{3}}$$

$$\dot{S}_{ij}^{(n)} = 2G^{(n)}\dot{e}_{ij} - \frac{S_{ij}^{(n)}}{\tau^{(n)}} - \frac{G^{(n)}}{G}\left[3\left(\frac{c}{a}\right)^{2}\frac{\dot{c}}{a}S_{ij} + \left(\frac{c}{a}\right)^{3}\dot{S}_{ij}\right]$$

# Fracture Modeling

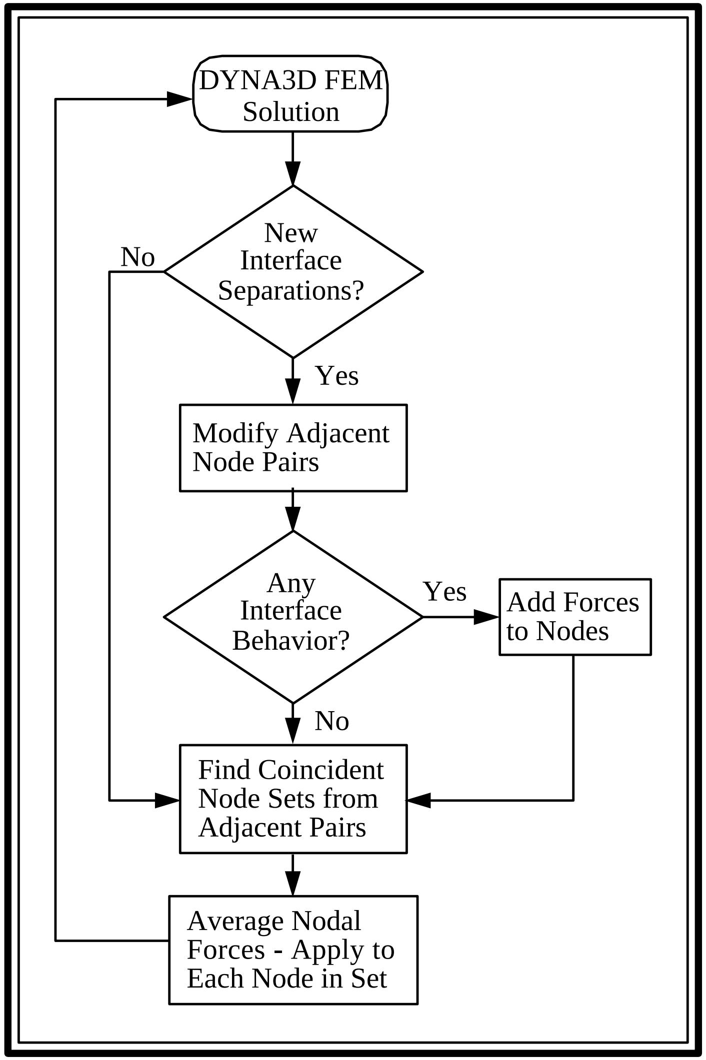
- Decompose Mesh
  - Add Nodes
  - Store Interface Data
  - Apply BC's





# Fracture Modeling

- Evaluate Fracture
  - Function of local variables
    - Nodal - acc., vel., disp., etc.
    - Element - $\varepsilon$, $\sigma$, material parameters, etc.
    - Time
    - Interface state variables
- Discontinuous Interface
  - Traction - Nodal force added to FEM solution
  - No Traction - (Semi)Independent surface



# Fracture Modeling

- Ensure Continuity

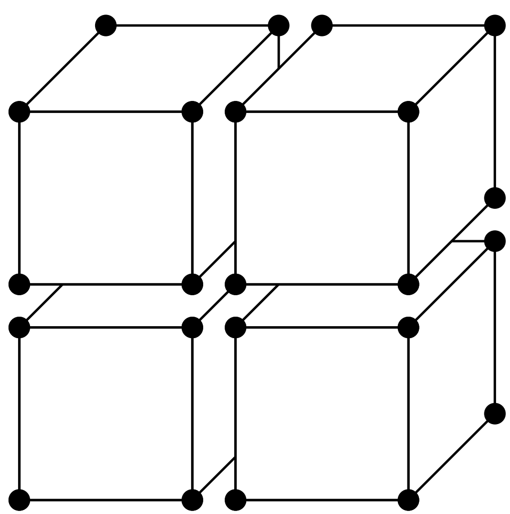
$$\vec{\vec{a}} = \vec{F} / M$$

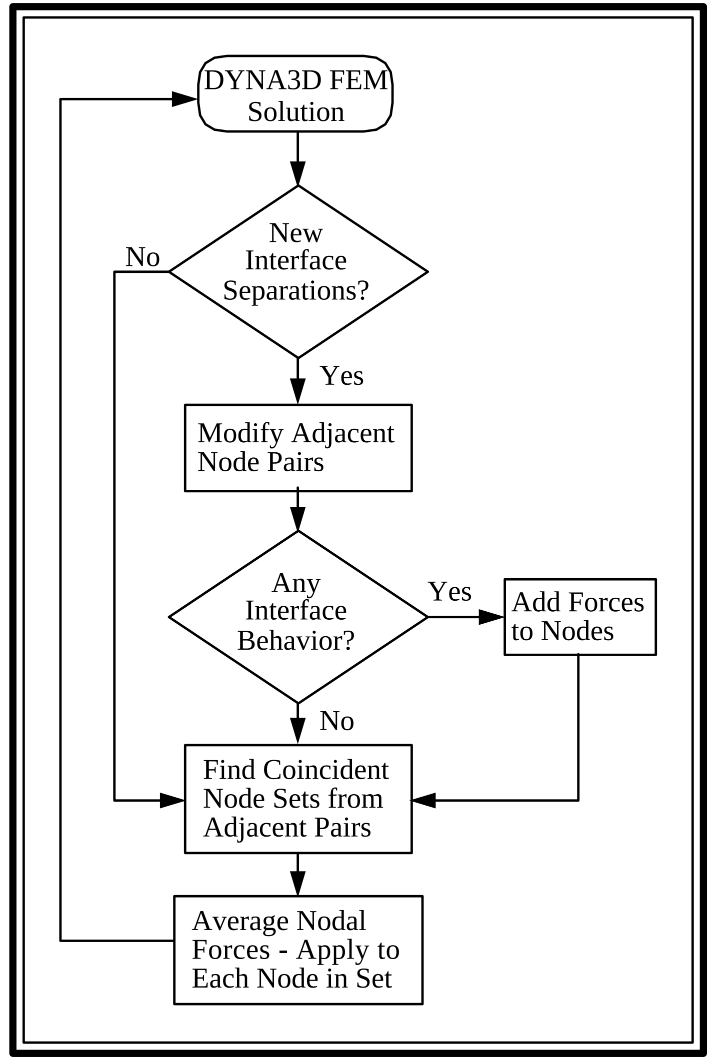
$$\vec{f}_i = \vec{\vec{a}} / m_i$$

where,

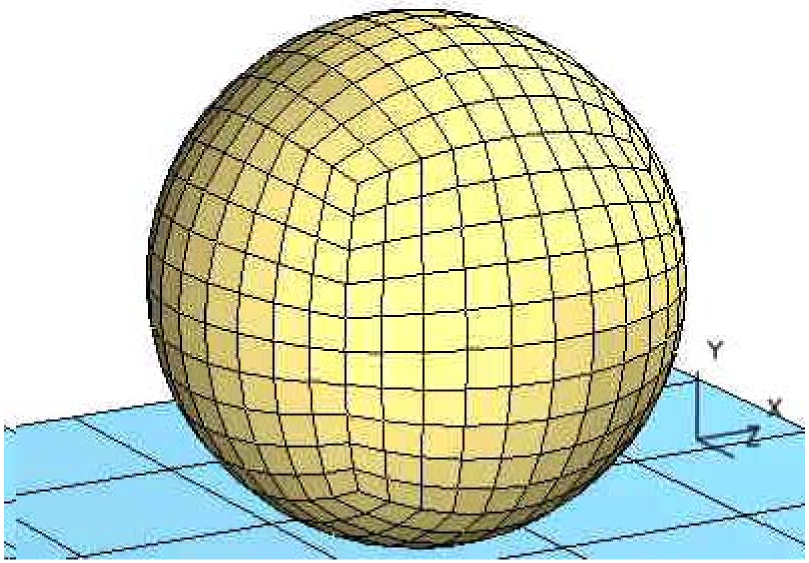
$$\vec{F} = \sum_i \vec{f}_i$$

$$M = \sum_i m_i$$

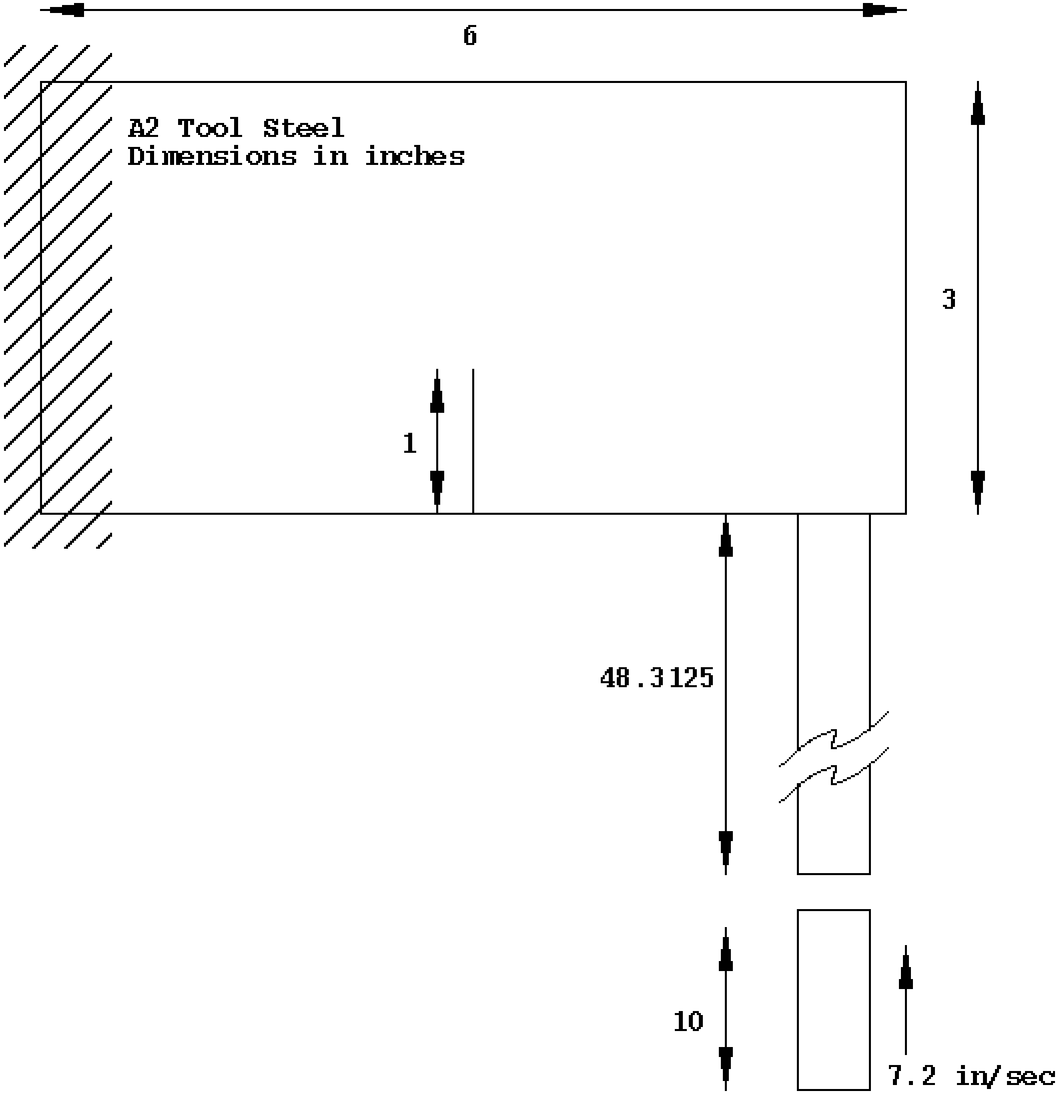




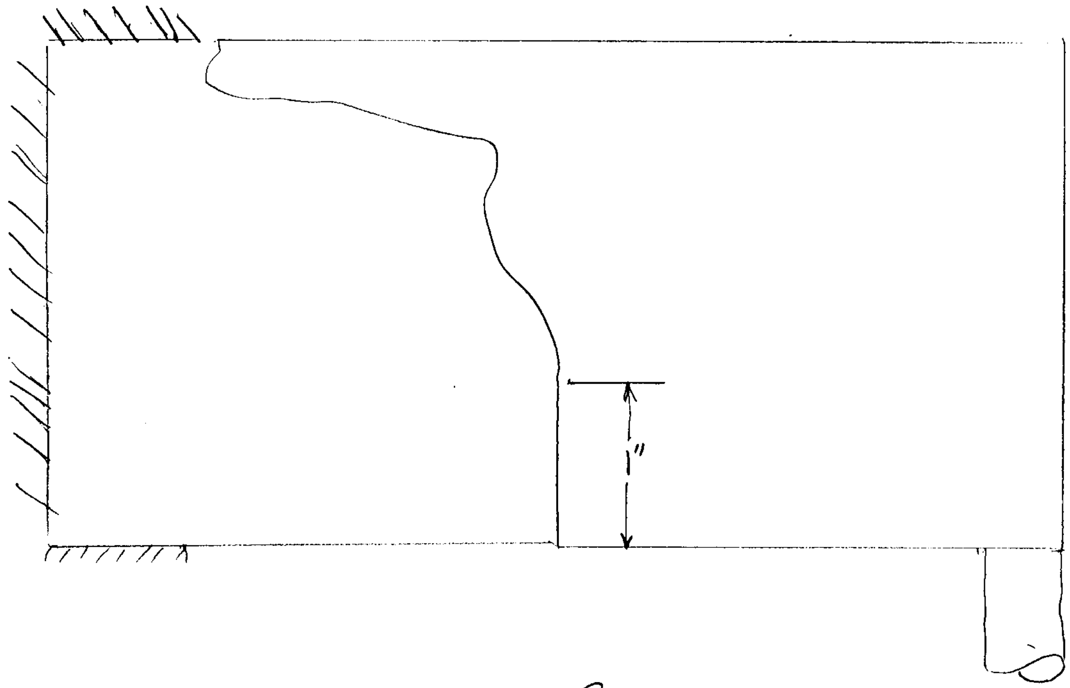
# Ceramic Ball



# Cantilever Impact



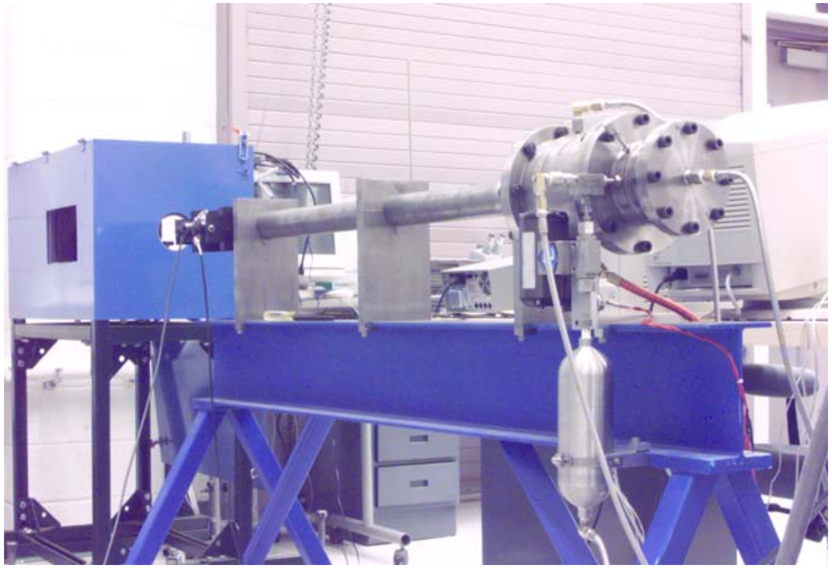
# Cantilever Impact

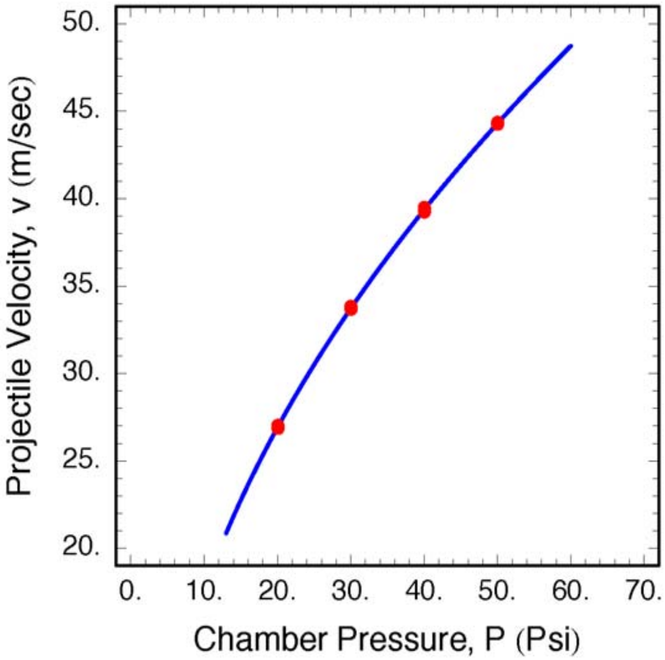


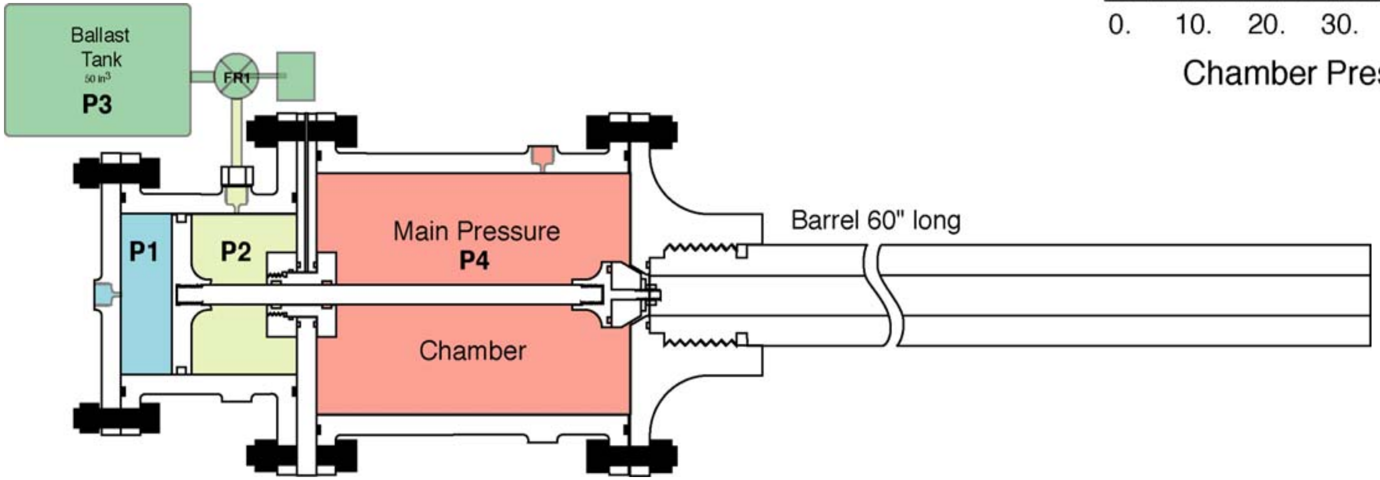

TRACING OF DYNAMIC FRACTURE SAMPLE
After Failure A-2 Steel. 6/11/98
Liu, Stout, Gerken, Smith

# Cantilever Impact

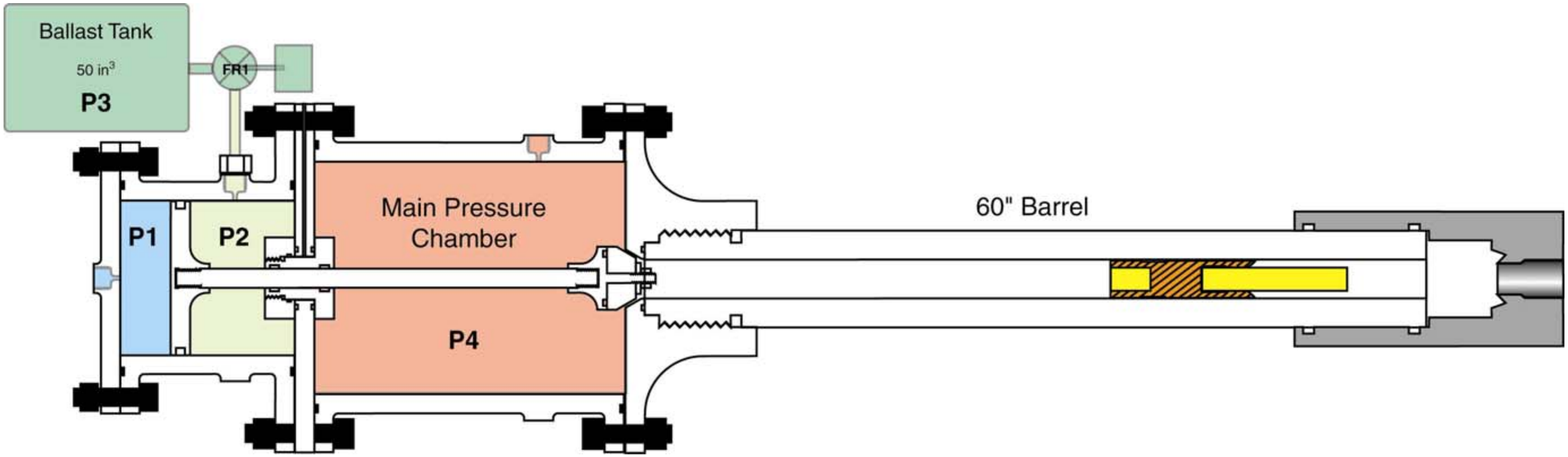
# Gas Gun Design & Operation

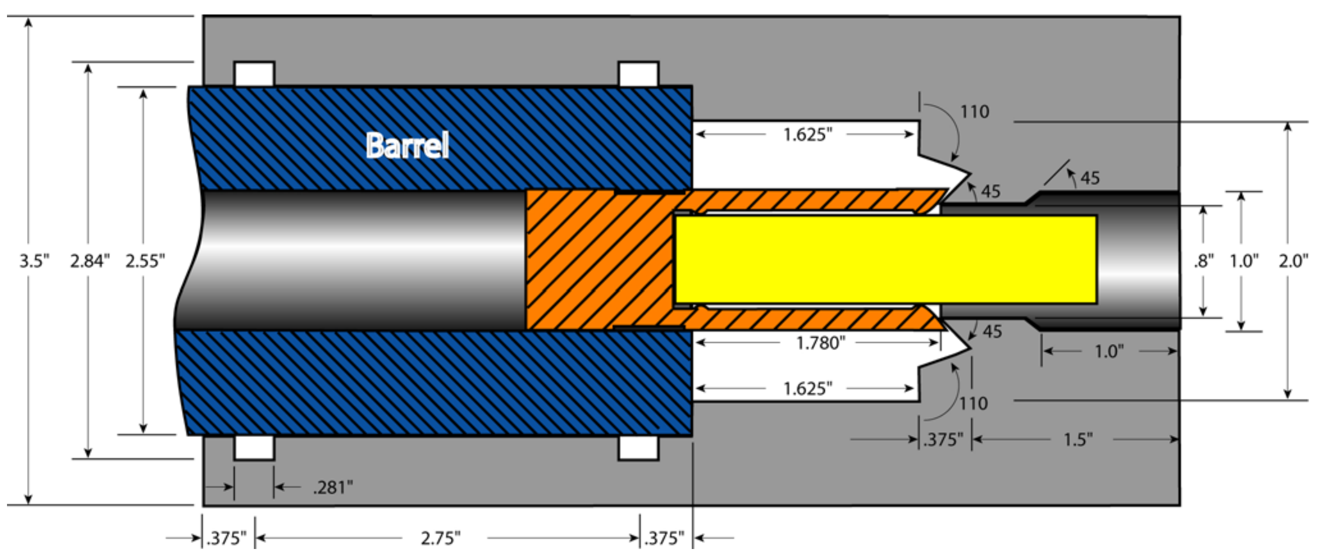




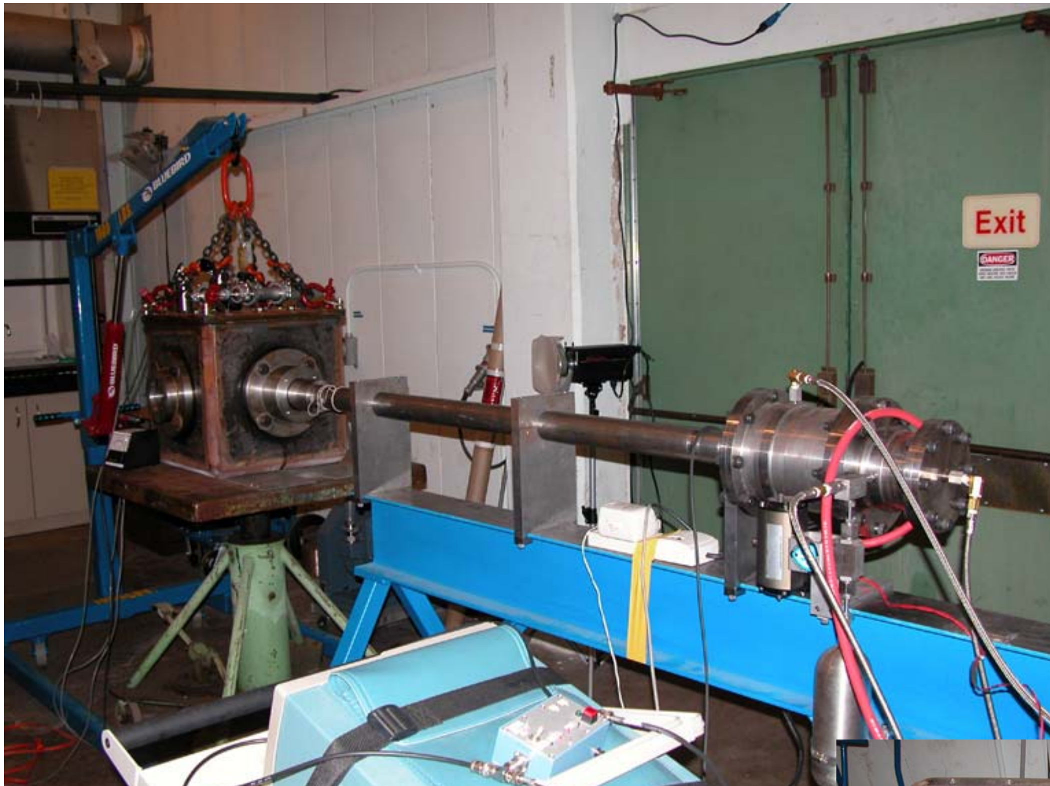



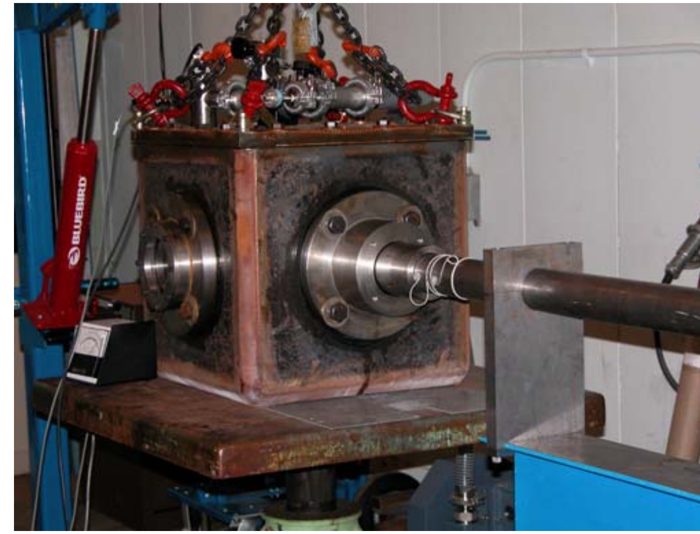
# Gas Gun & Sabot Stripper



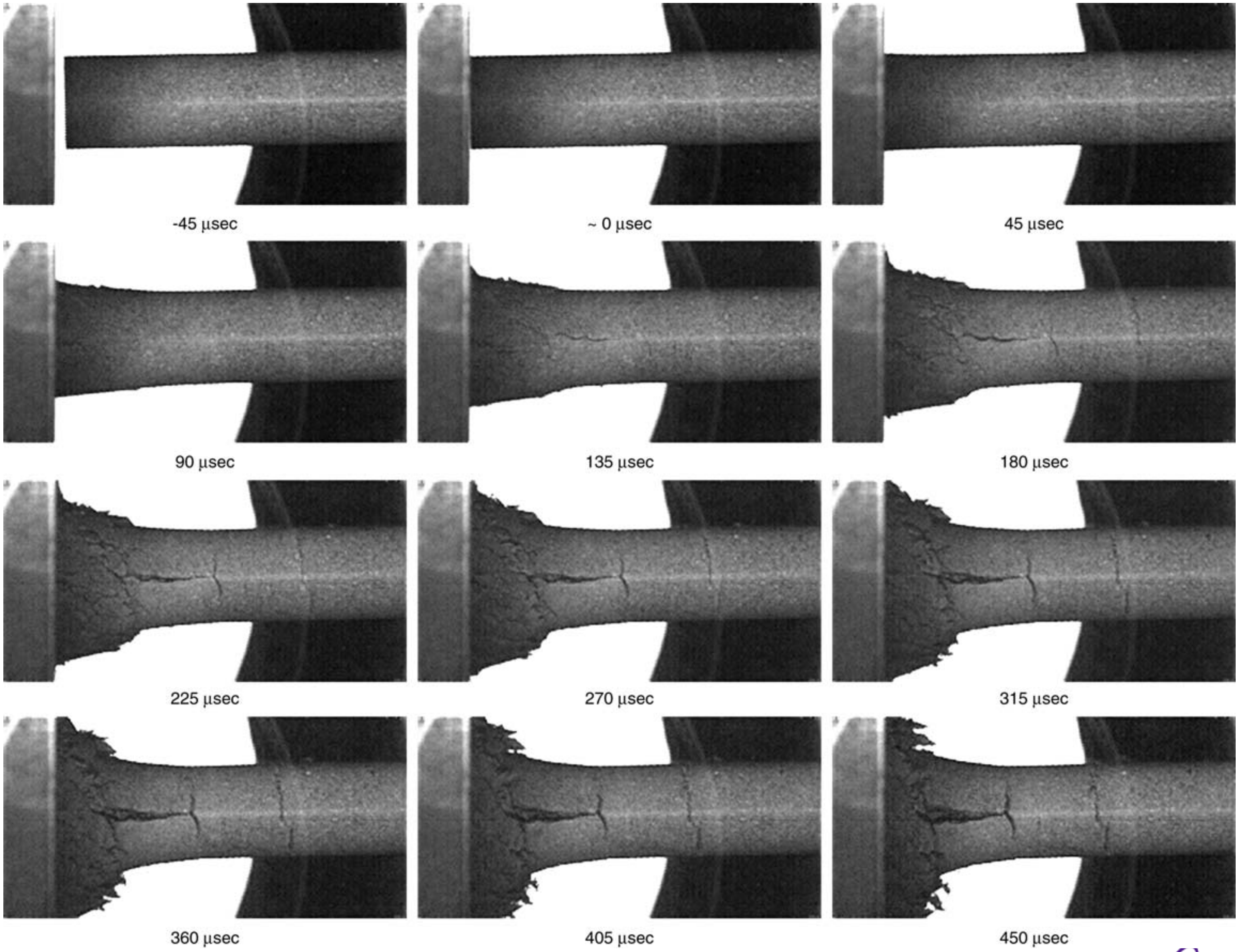
# Gas Gun, Boom Box, & Anvil Setup



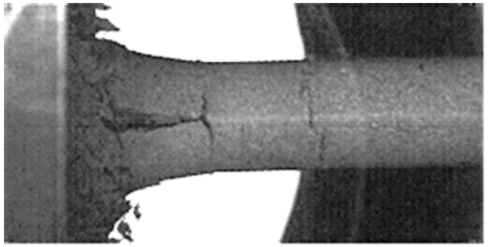



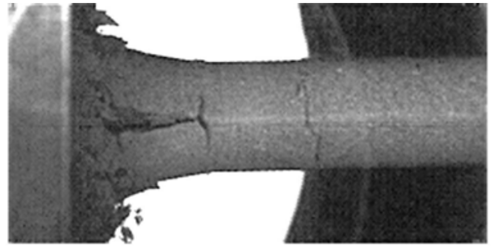
**The boom box was certified for handling 45 grams of HE**

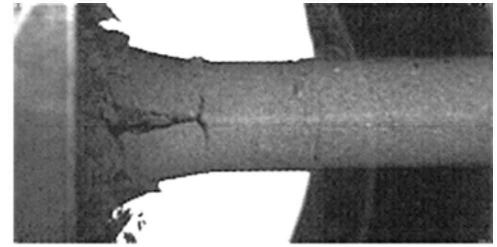
# Taylor Cylinder Impact of Mock 900-21 at 38.9 m/sec (1)

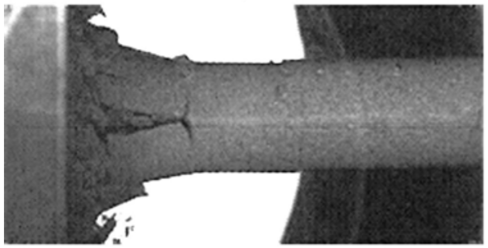


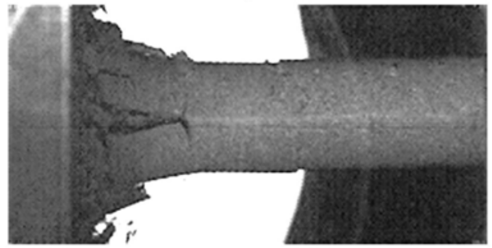
# Taylor Cylinder Impact of Mock 900-21 at 38.9 m/sec (2)

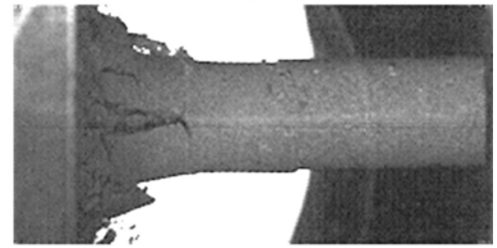

495 μsec

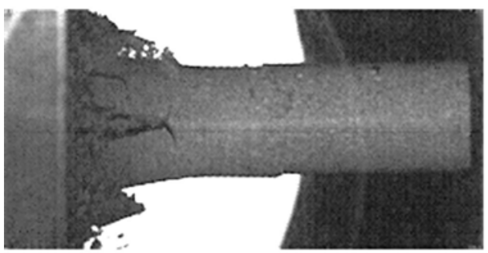

540 μsec

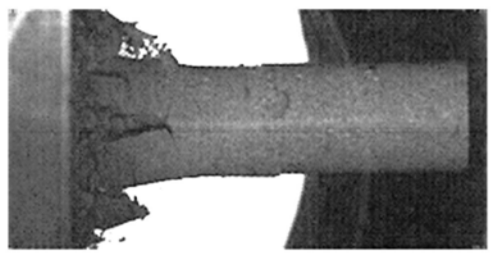

585 μsec

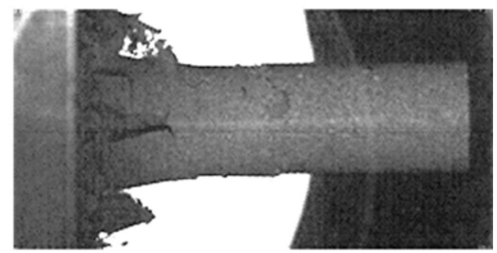

630 μsec

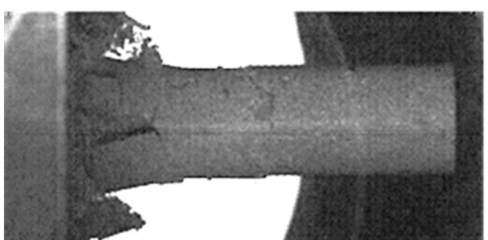

675 μsec

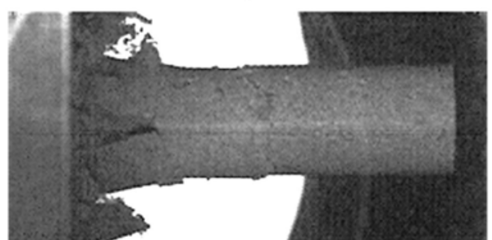

720 μsec

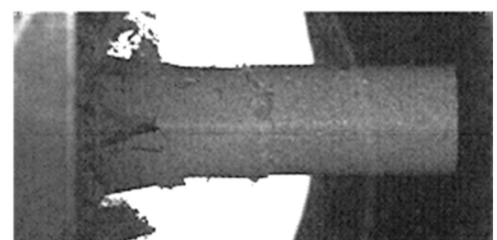

765 μsec


810 μsec


855 μsec

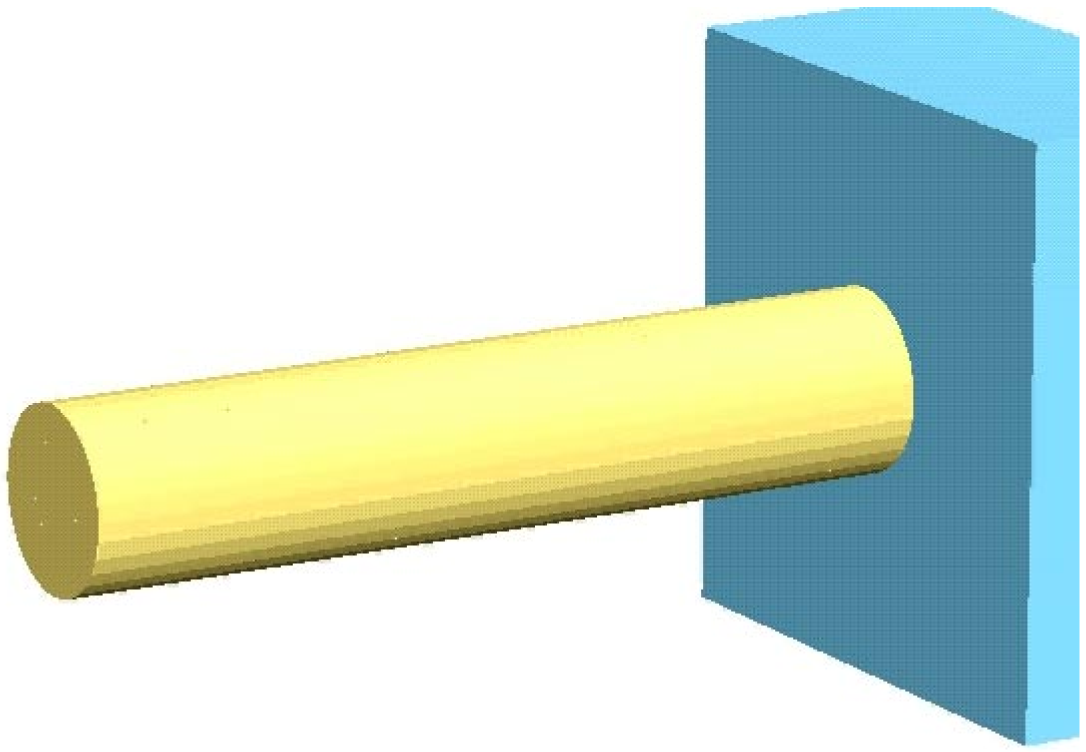

900 μsec


945 μsec

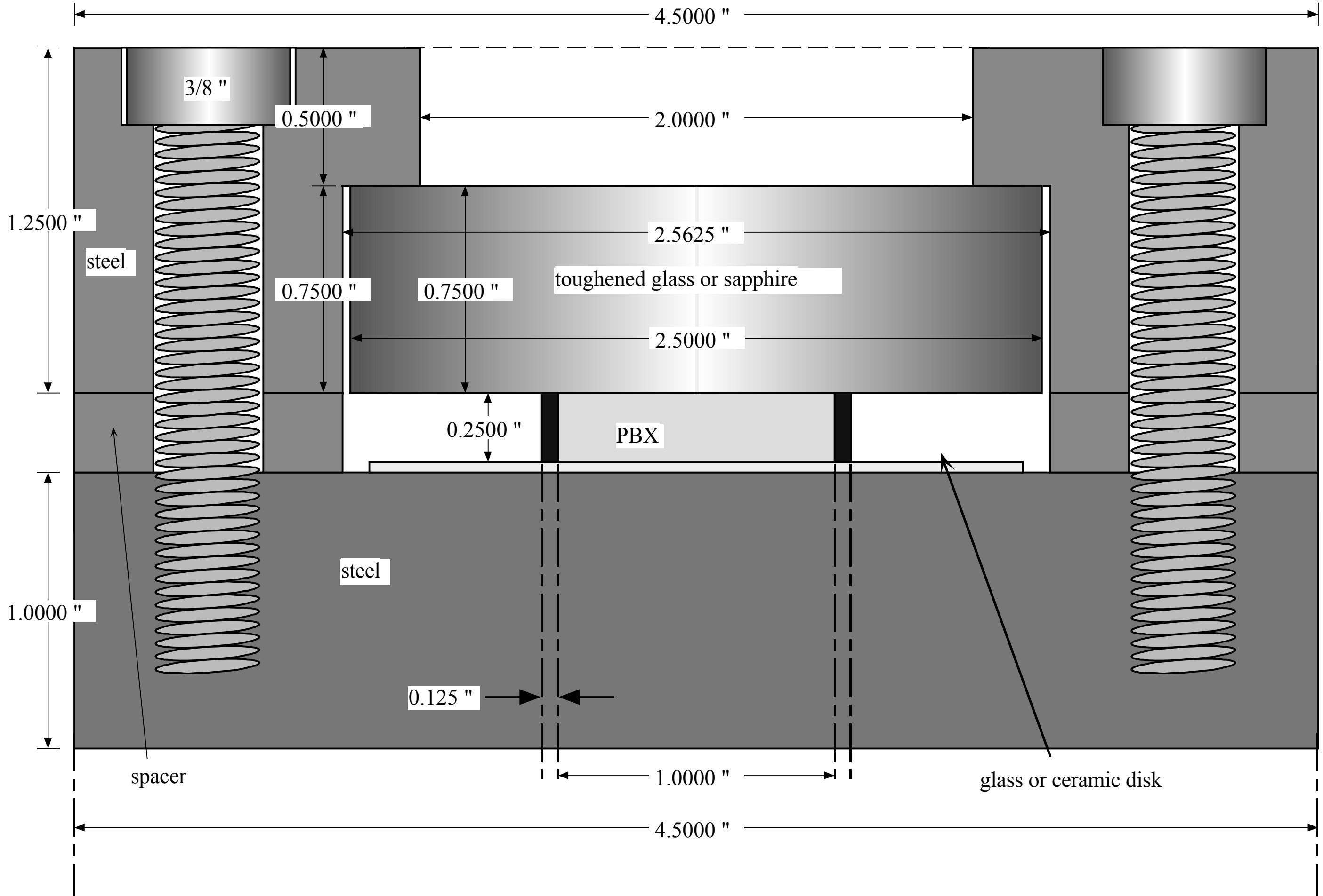

990 μsec

# Taylor Cylinder impact

# Taylor Impact of PBX 9501



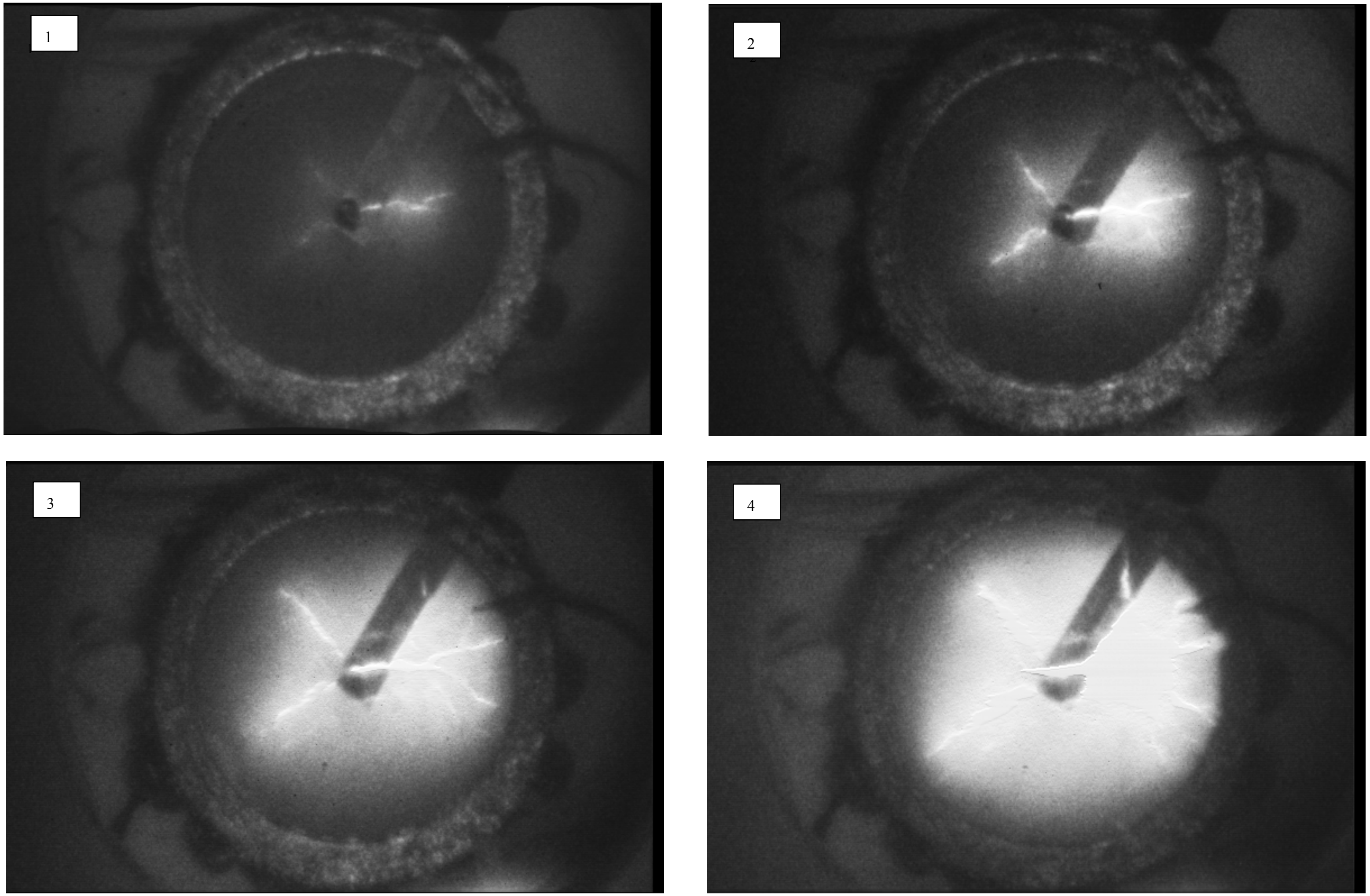
# Mechanically Coupled Cook off



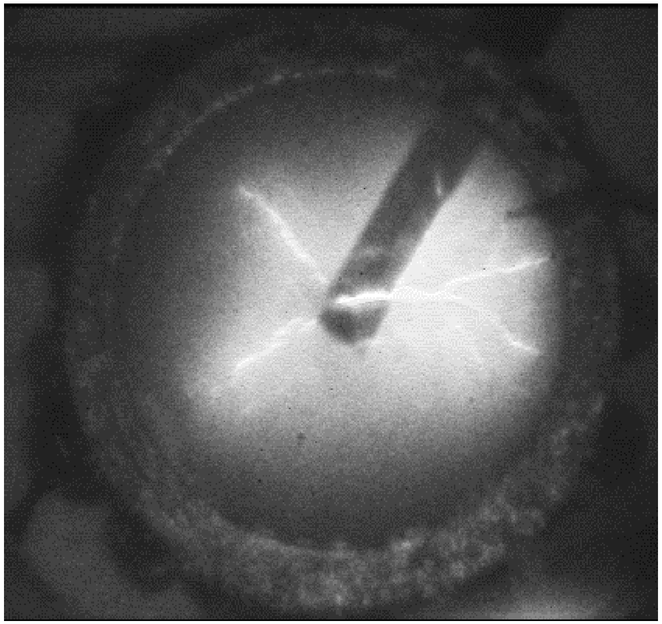

**Side view**

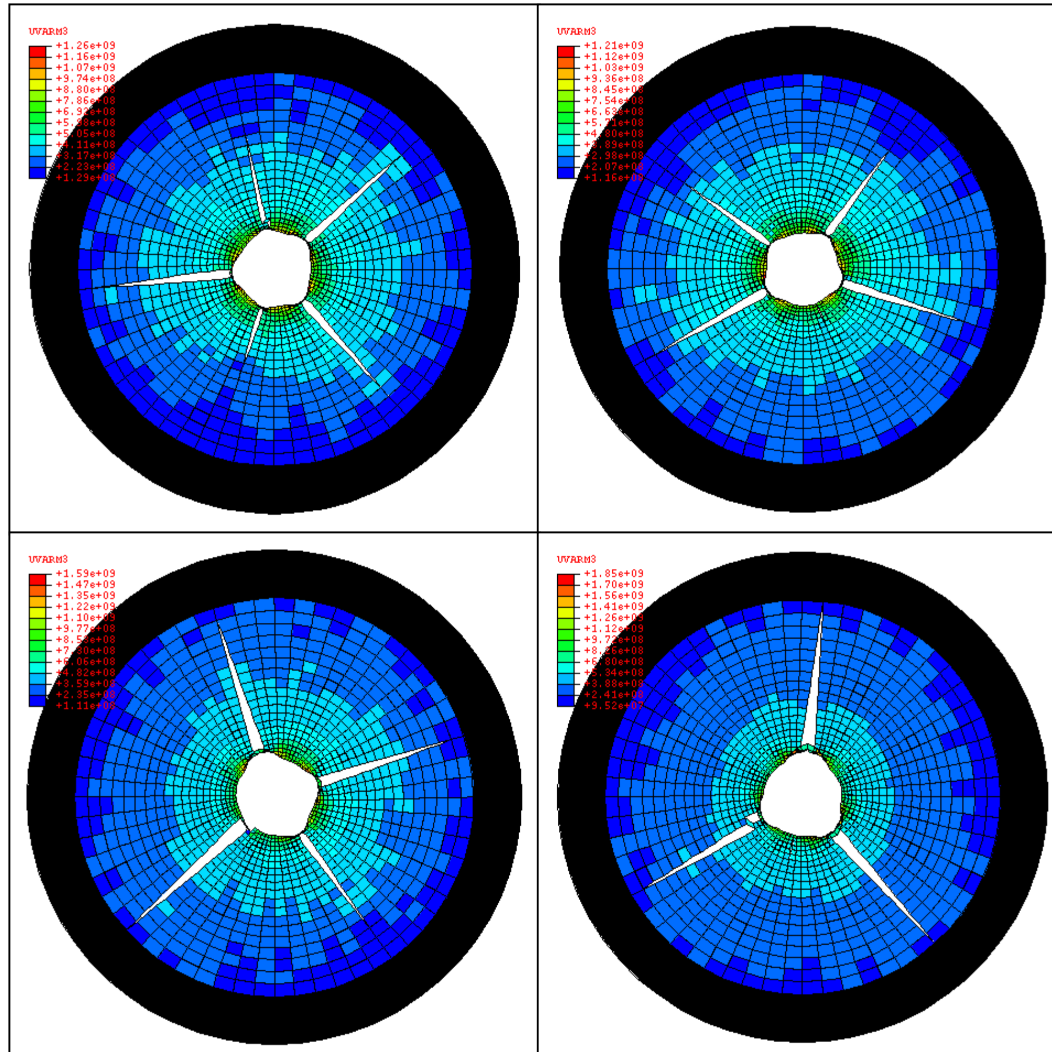
Scale: x2

# Mechanically Coupled Cook off



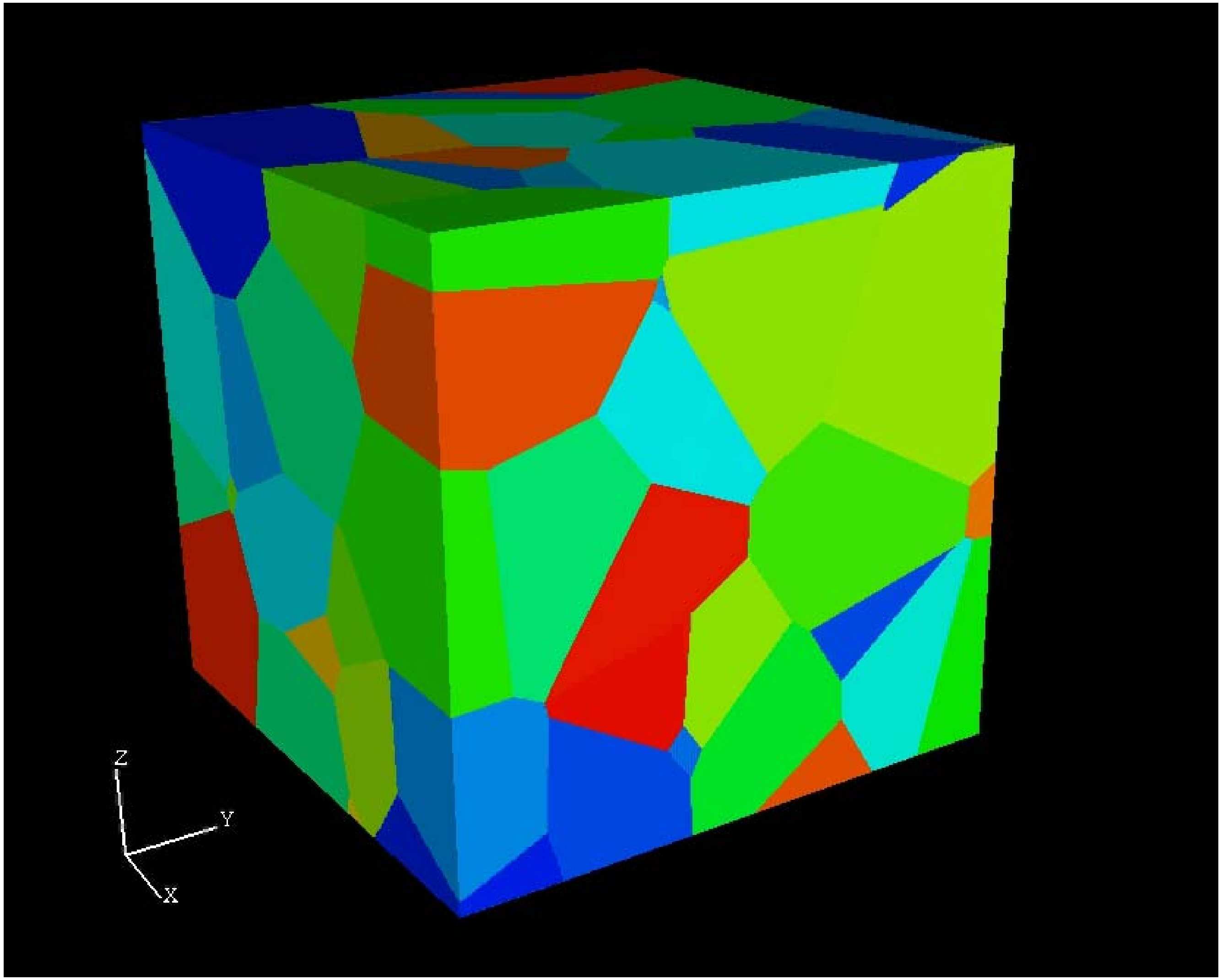
# Mechanically Coupled Cook off





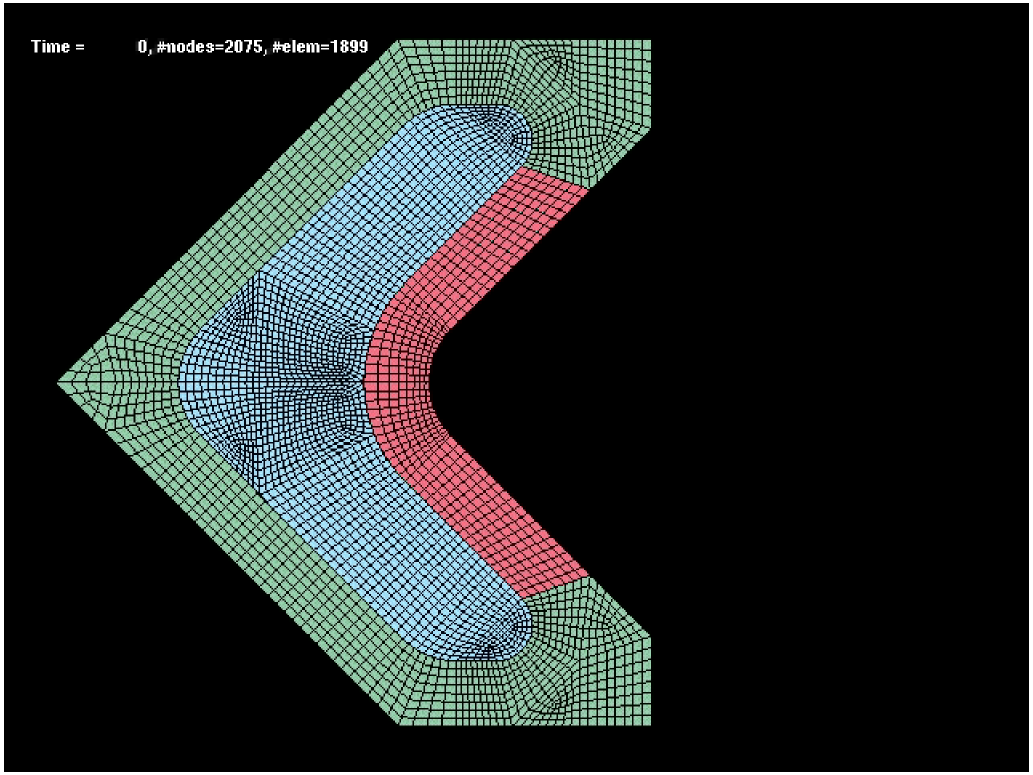

Using randomized failure criteria, the simulations show qualitative agreement with experimental results

# Crystal Structure



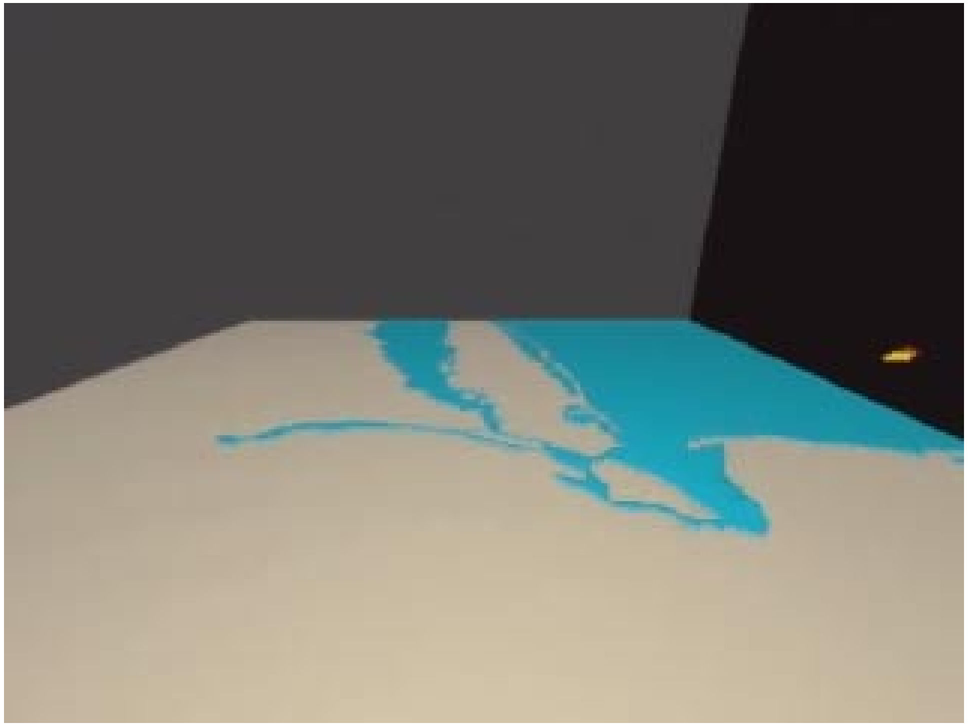
# Earth Penetrating Weapons

# Some cool stuff I found on the web



# Some cool stuff I found on the web

- SNL - Comet impact

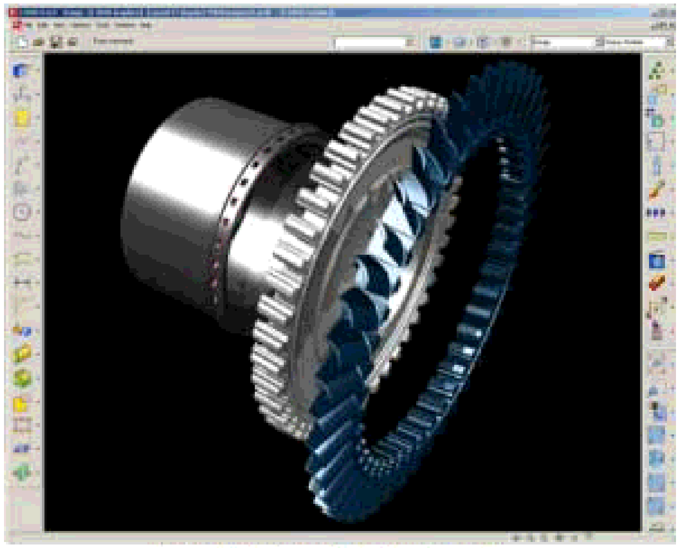


## The reality of Finite Element modeling

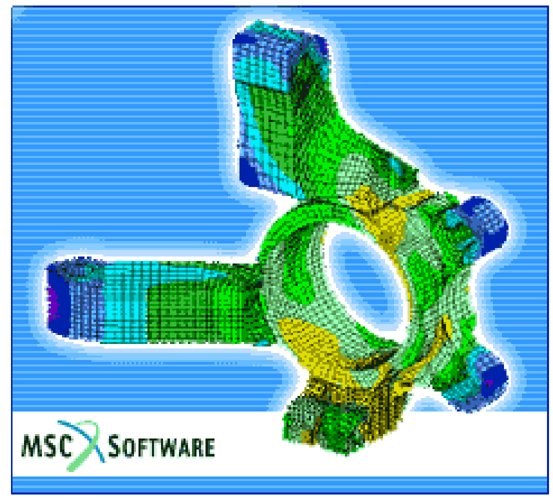
- The steps to modeling
  - Get a mesh of your system
  - Define the properties
    - Loading
    - Boundary conditions
    - Material Behavior
    - Body interaction
  - Solve the model
  - View the results
  - Decide they are wrong and start over

# Get a mesh

- Canned packages



New Windows Interface for
I-DEAS

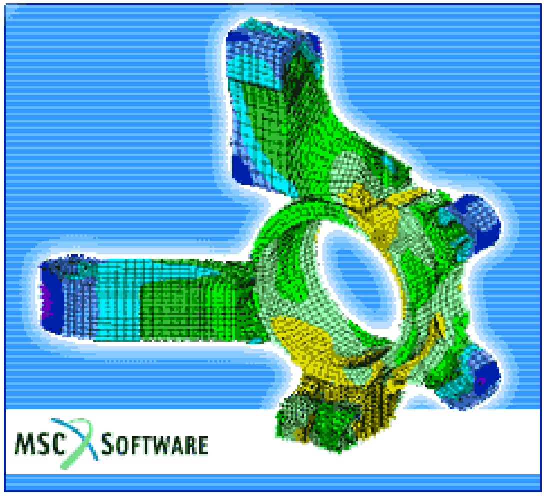


## Solve the model

- Integrated solution environments

ABAQUS

Hibbitt, Karlsson & Sorensen, Inc.

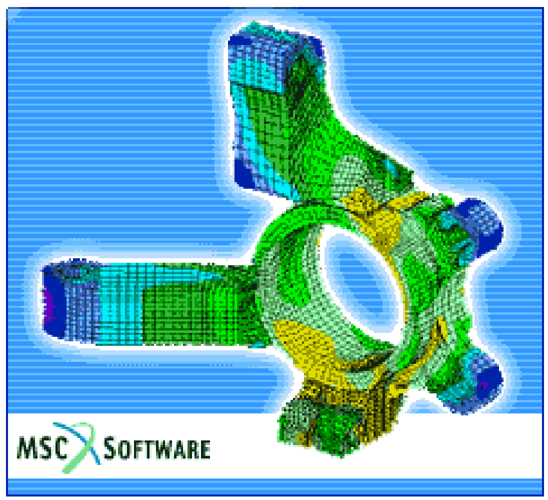


## Solve the model

- Specilaized FE codes
  - DOE
    - LLNL - DYNA3D, NIKE3D, TOPAZ
    - SNL - PRONTO3D, JAS, SALINAS, COYOTE
  - Other government codes
  - Research developed codes

- Integrated solution environments

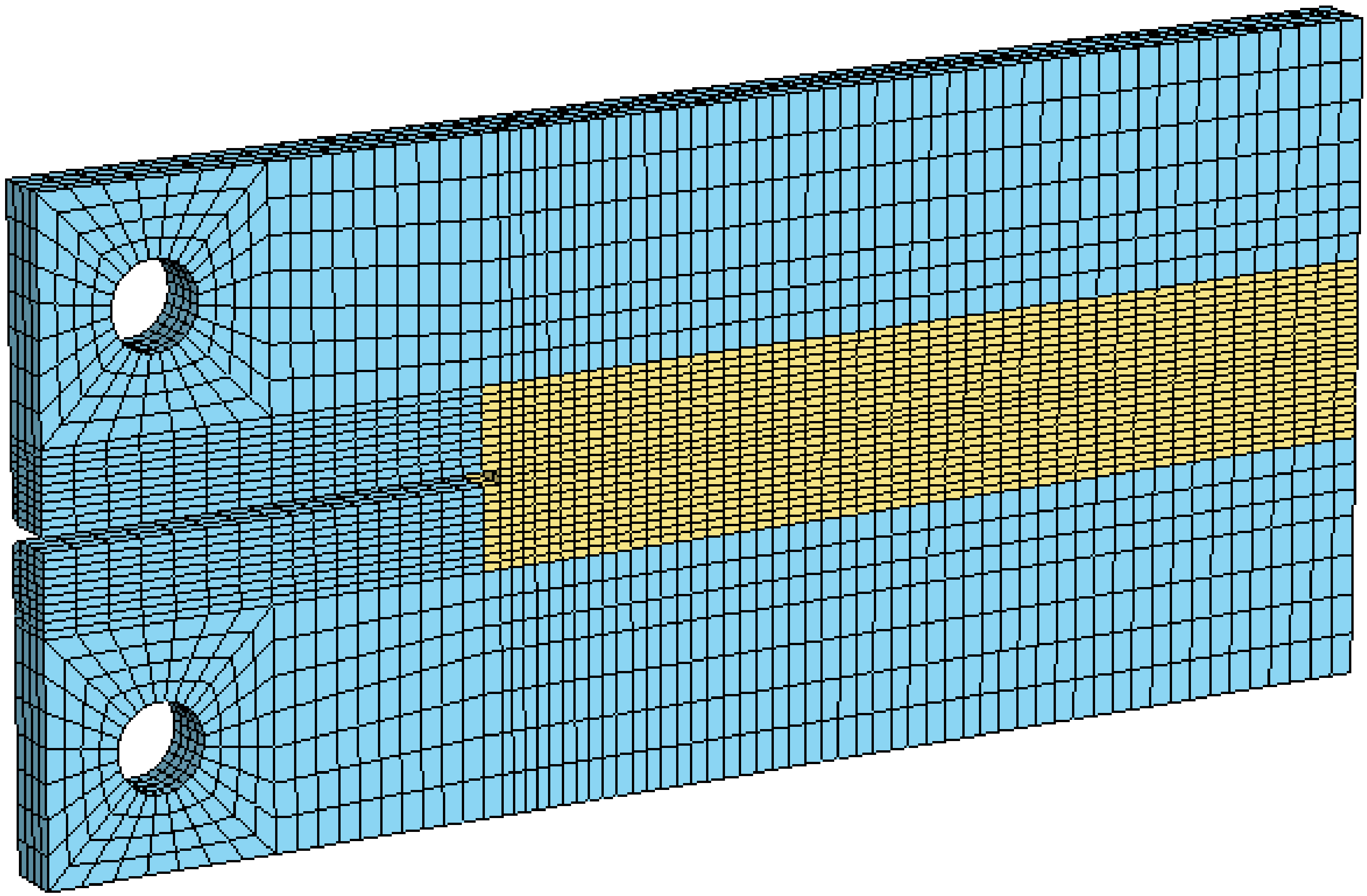
ABAQUS Hibbitt, Karlsson & Sorensen, Inc.

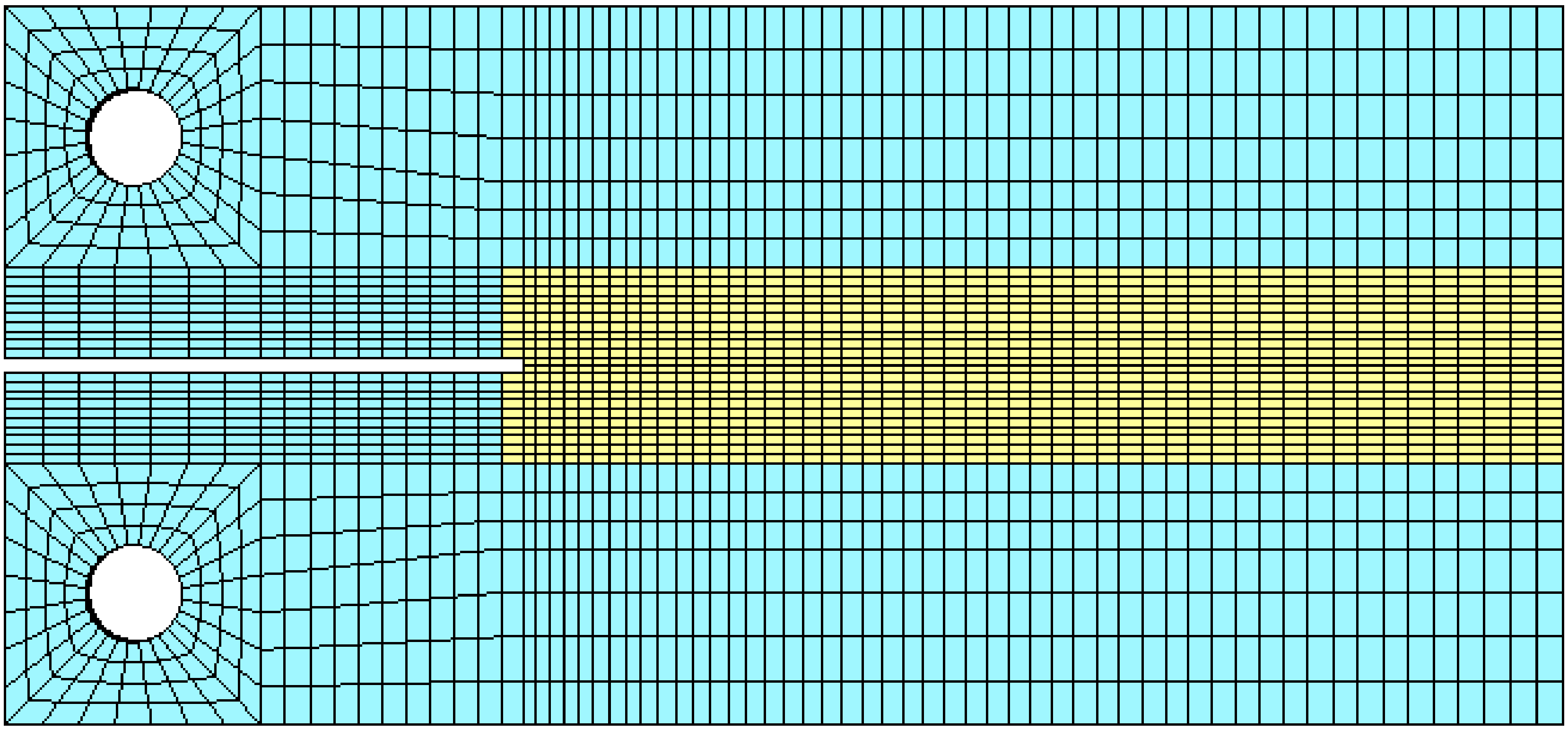


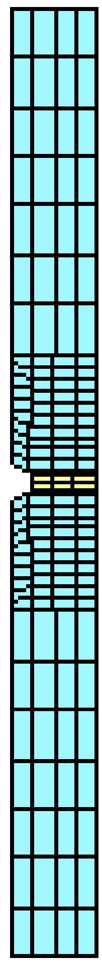
## View the results

- Many codes (integrated environments or not) have post-processors.
- Stand alone post-processors are available. (*e.g.* Ensight)
- Sometimes simply *looking* at the results is not enough.

- Matching the relevant geometry
- Good element shape
- Automatic meshing



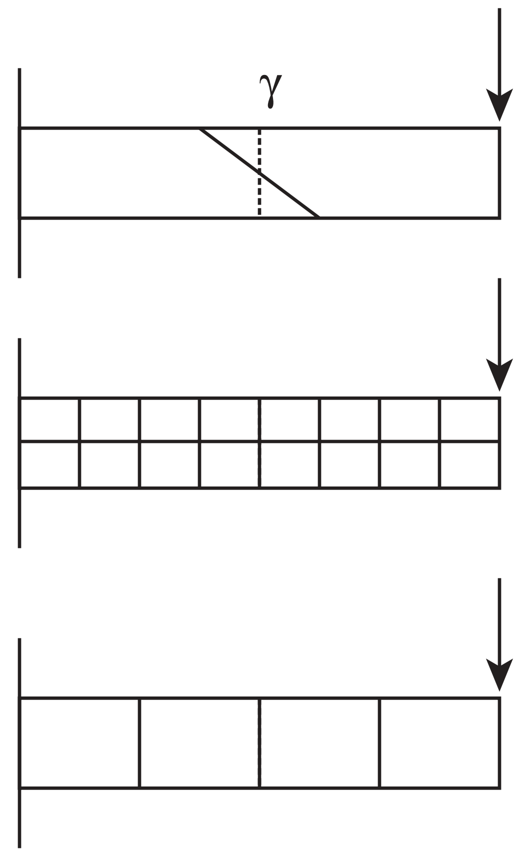




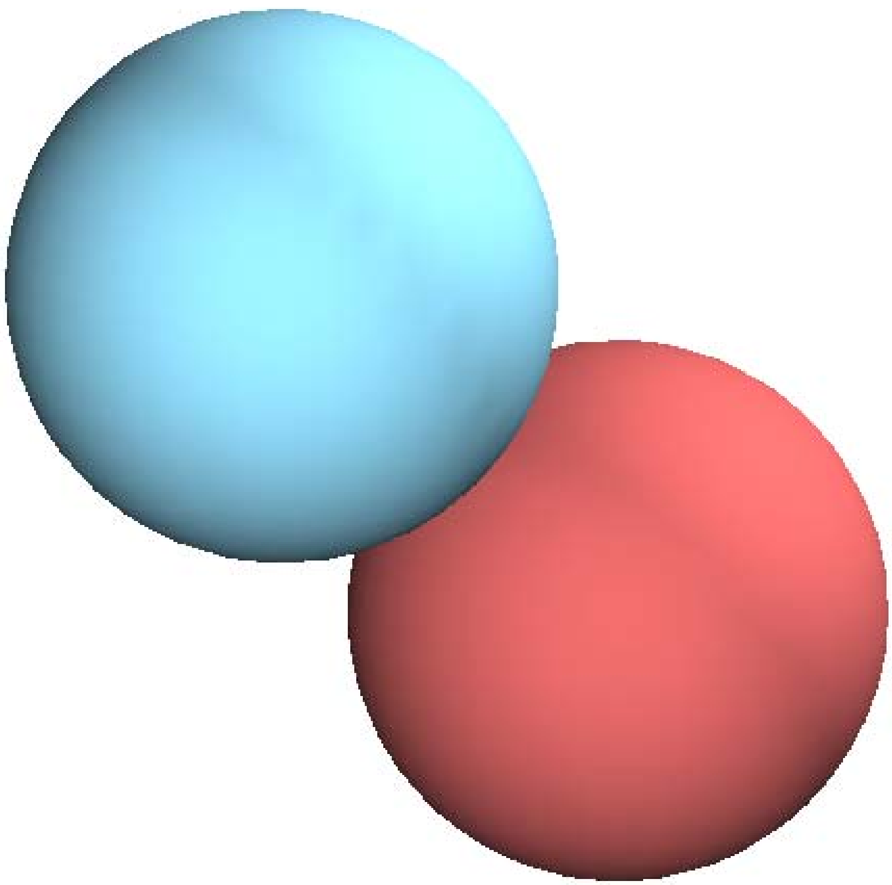
$$u^* = \sum_{i=1}^{n} N_i a_i$$

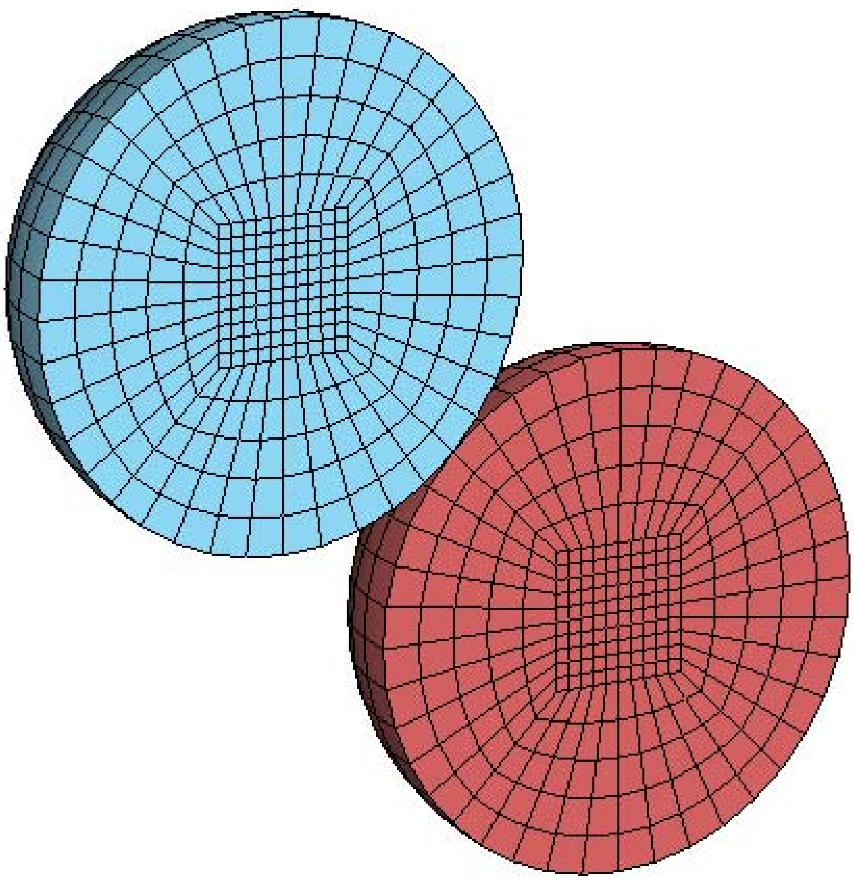
- linear
- quadratic
- cubic, ...

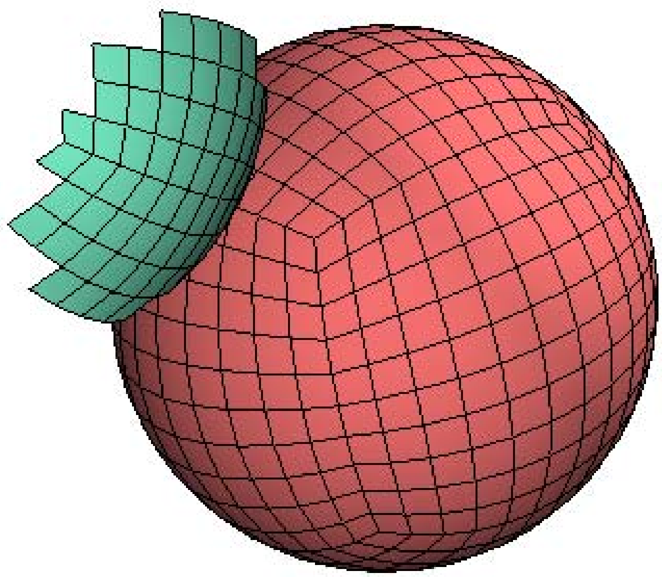
$$\epsilon = \frac{\partial u}{\partial x}$$

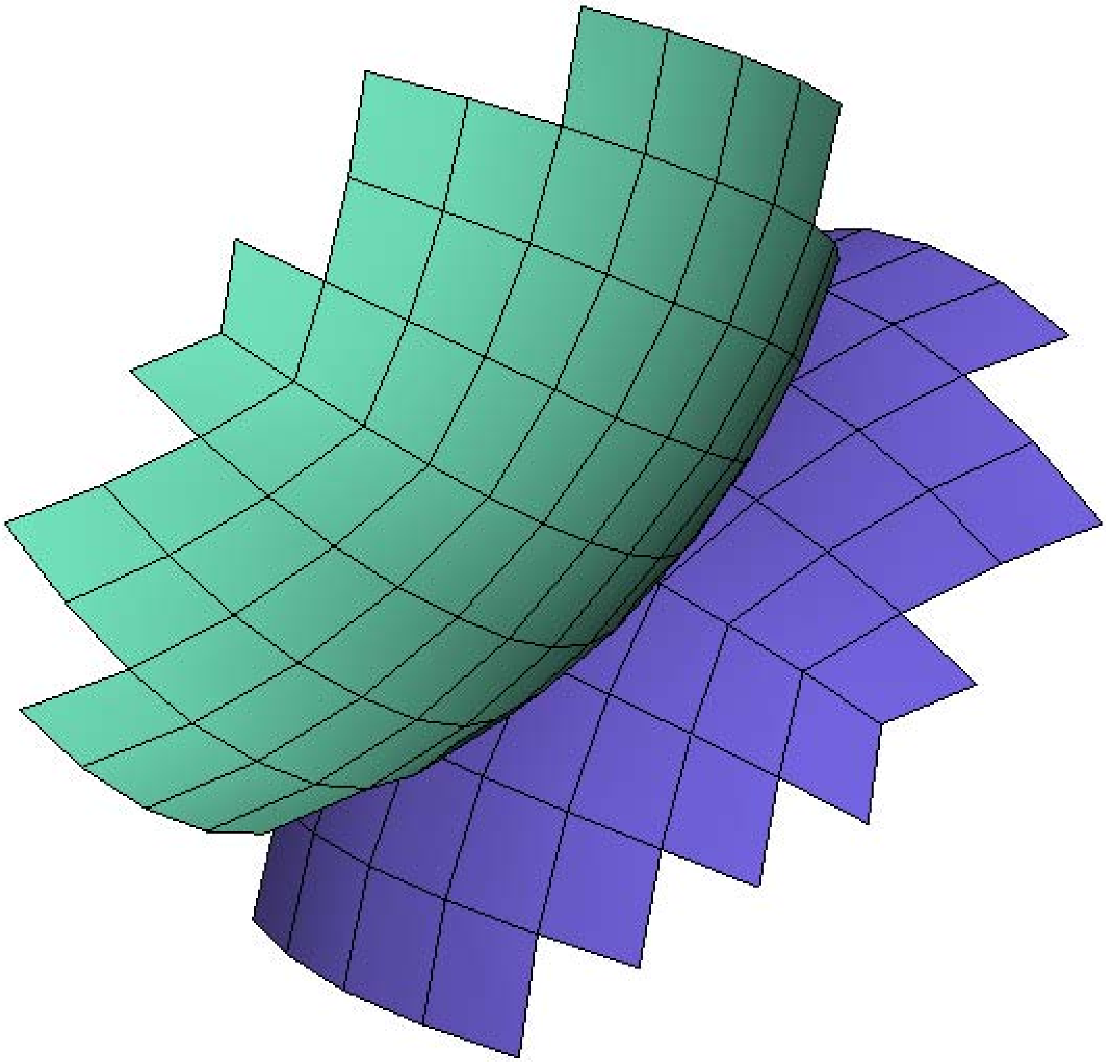


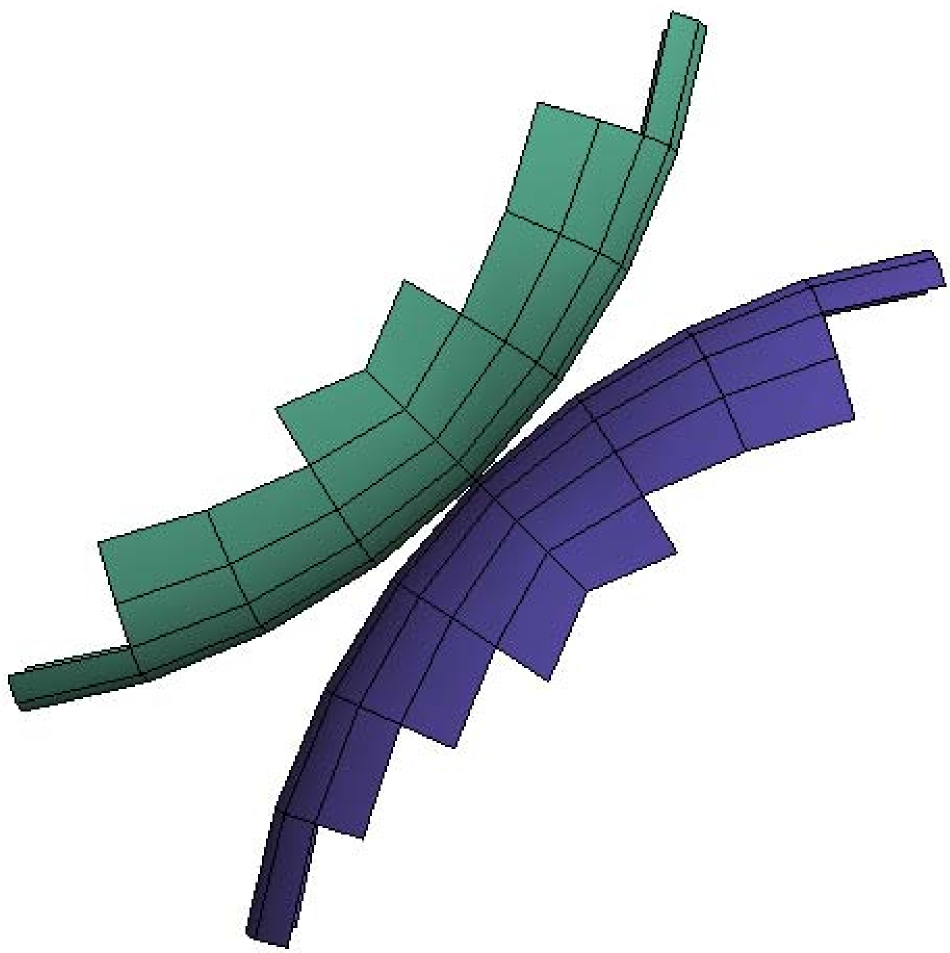
- Shape
  - 3D, Brick, Tetrahedral
  - 2D, Quadrilateral, Triangular
- Integration
  - Fully integrated
  - Reduced integration
  - Pressure integration
- Distortion



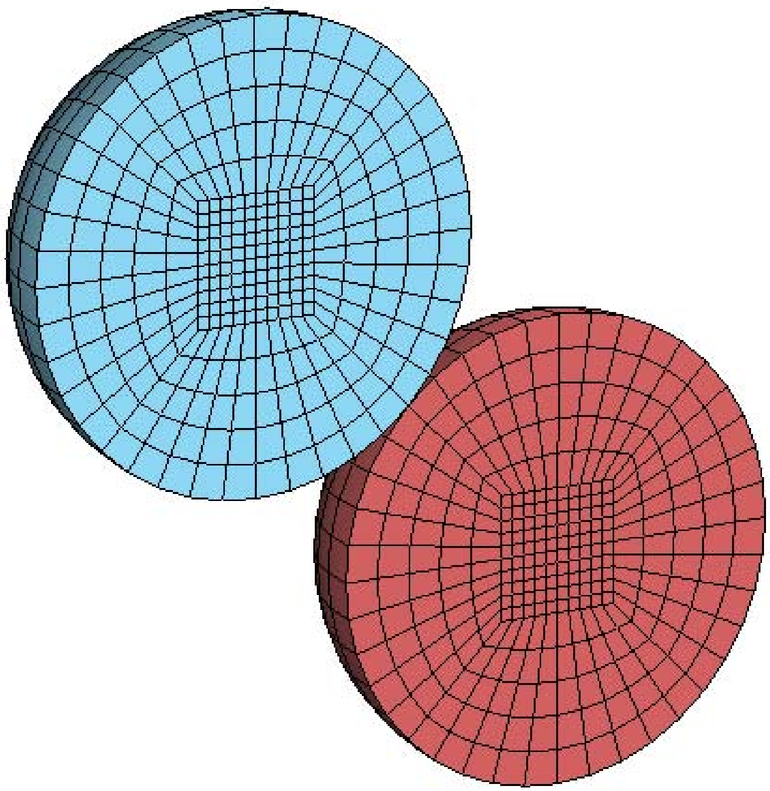


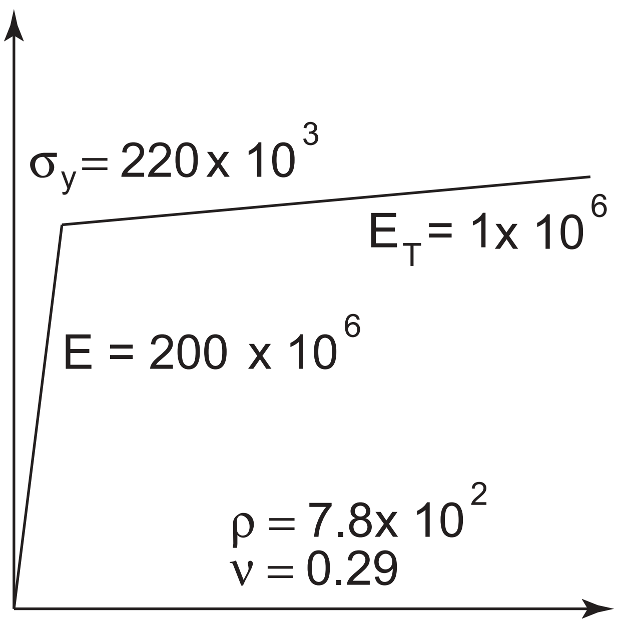






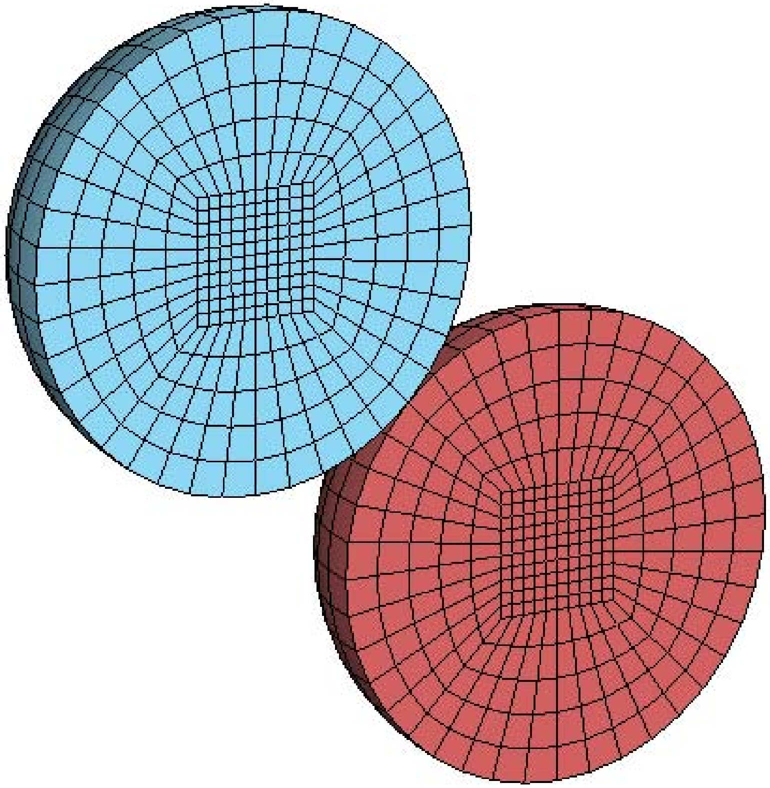
# Material





$\sigma_y = 220 \times 10^3$
$E_T = 1 \times 10^6$
$E = 200 \times 10^6$
$\rho = 7.8 \times 10^2$
$\nu = 0.29$

# Material/timing



$$c_l = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{200 \times 10^6}{7.8 \times 10^2}} = 500$$

$$c_l(steel) \approx 5000$$

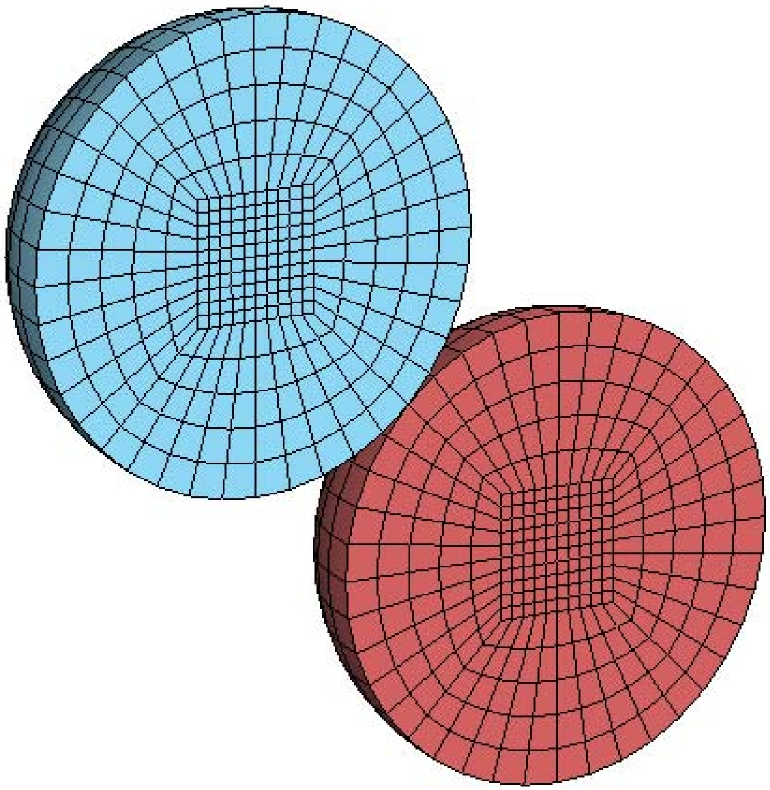
$$l_{min} \approx 2 \times 10^{-3}$$

$$\Delta t_{min} \approx \frac{2 \times 10^{-3}}{500} = 4 \times 10^{-6}$$

$$\Delta t_{actual} = 2.5 \times 10^{-6}$$

$$t_{sim} = 2 \times 10^{-2}$$

$$inc = \frac{2 \times 10^{-2}}{2.5 \times 10^{-6}} = 8000$$



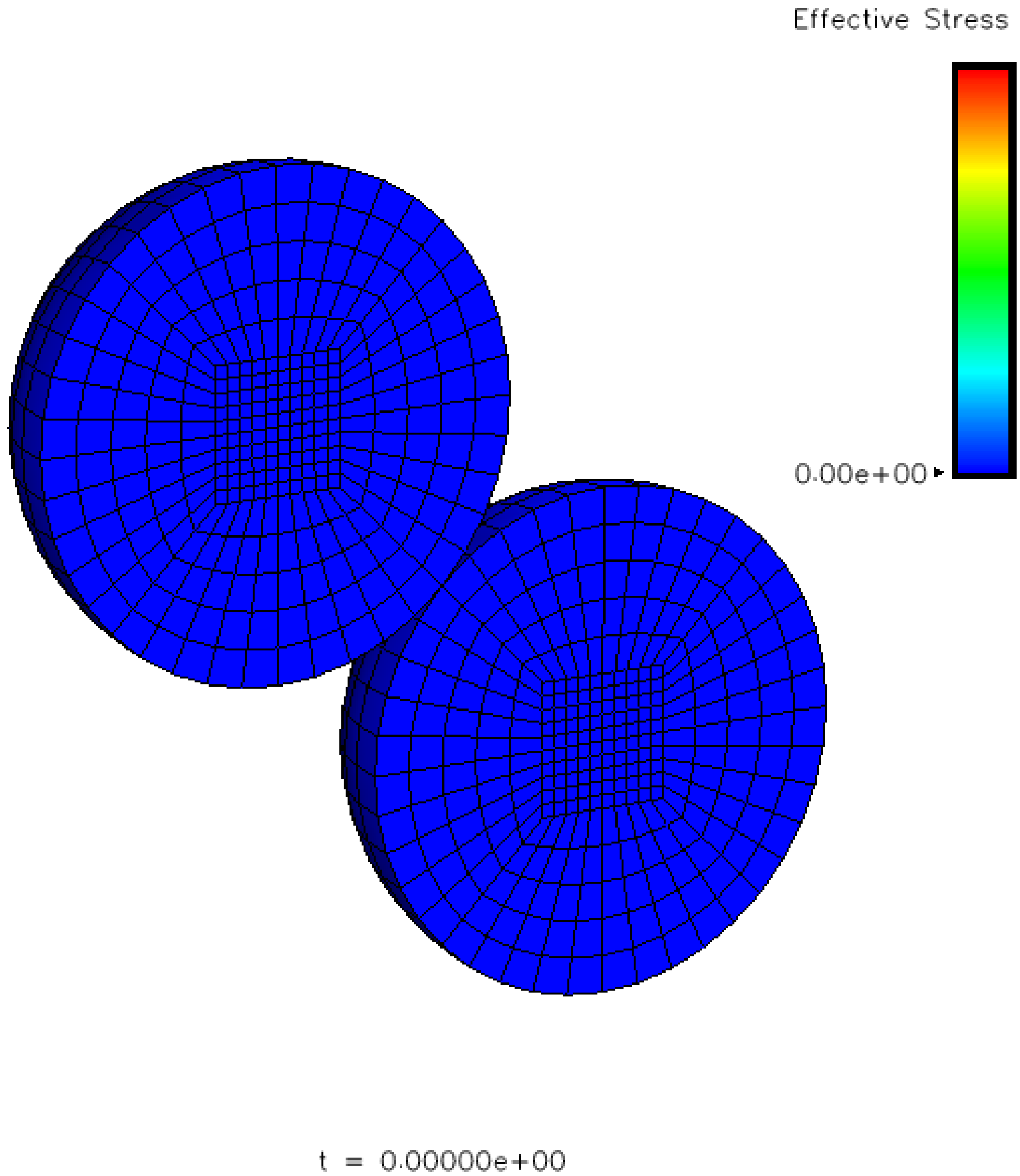
8000 elements

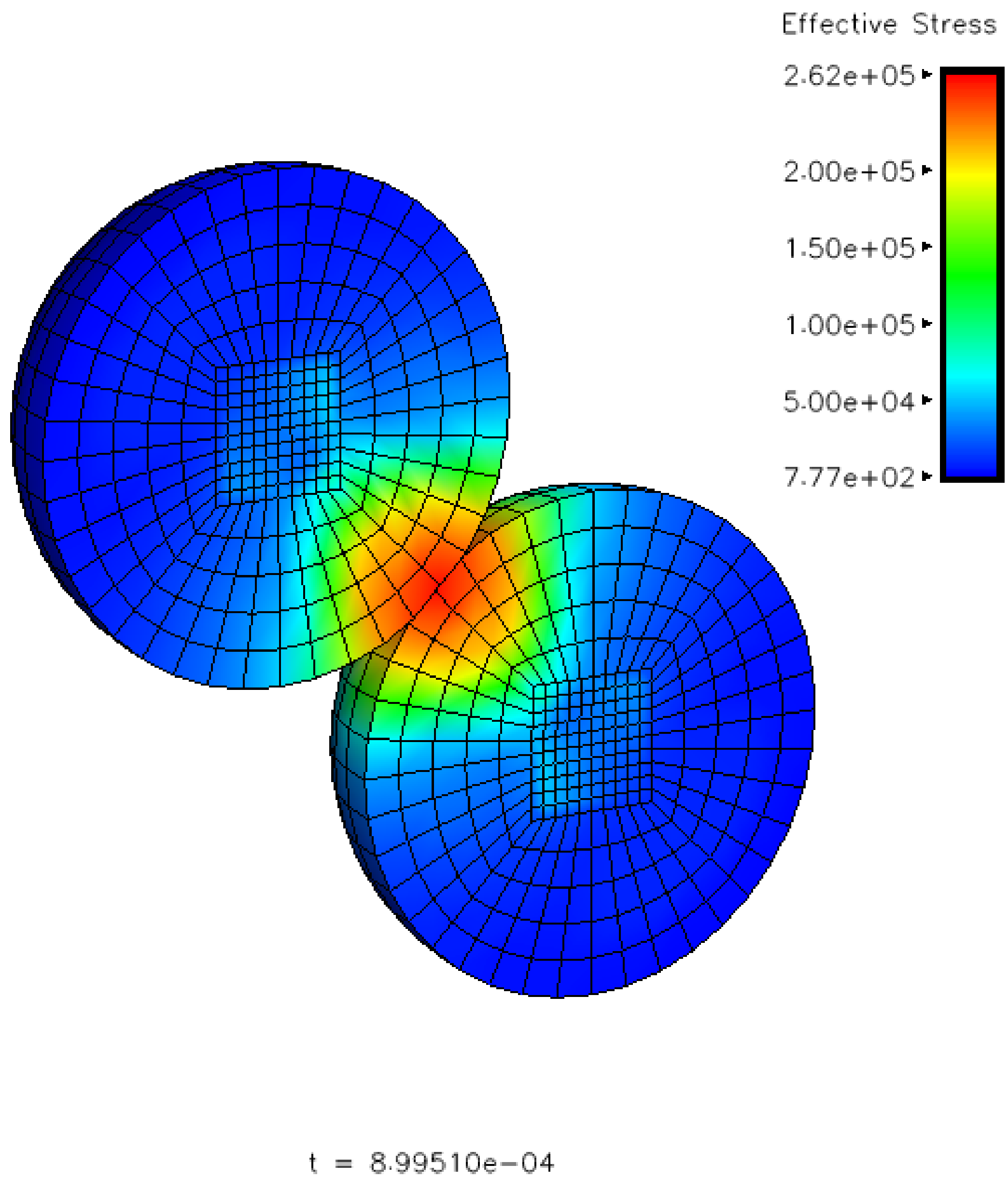
8682 nodes $\longrightarrow$ 26046 DOF

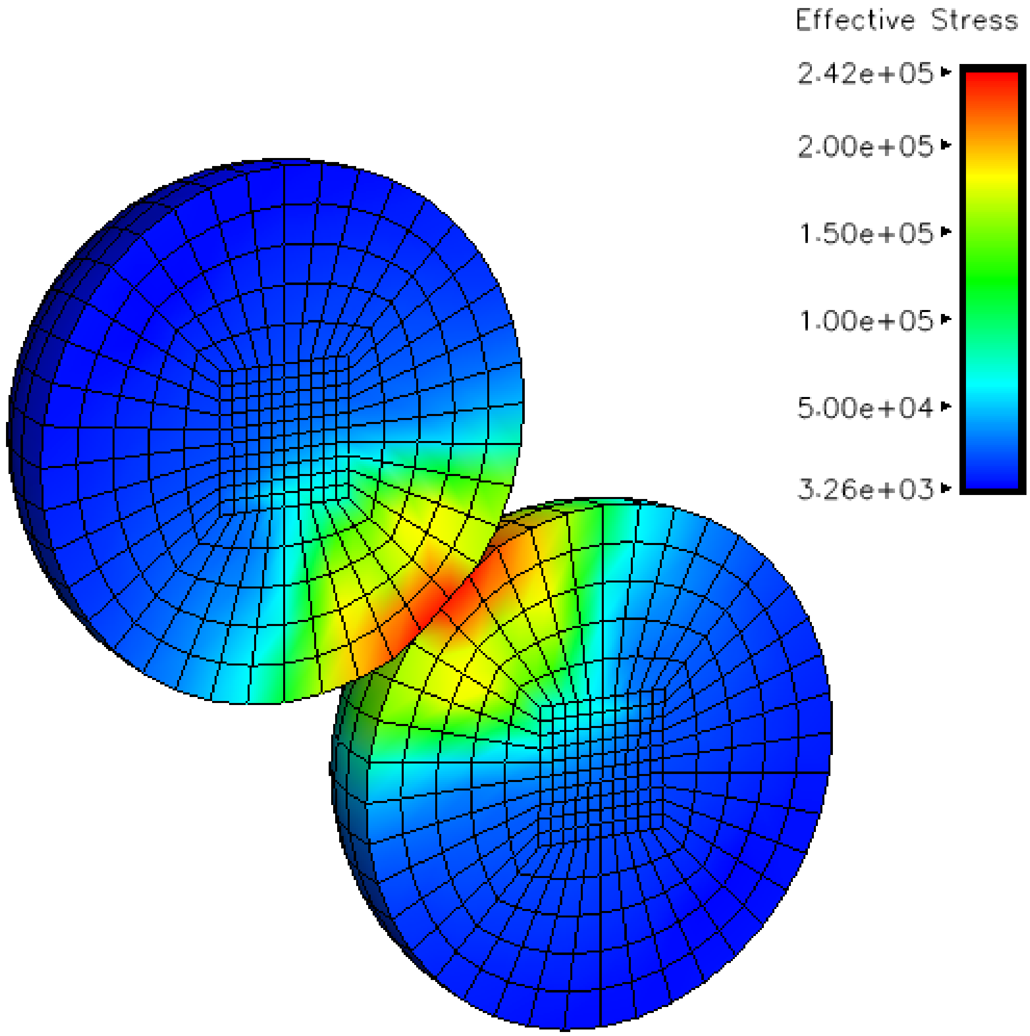
SGI IRIX 195 MHz

17 Minutes of computation

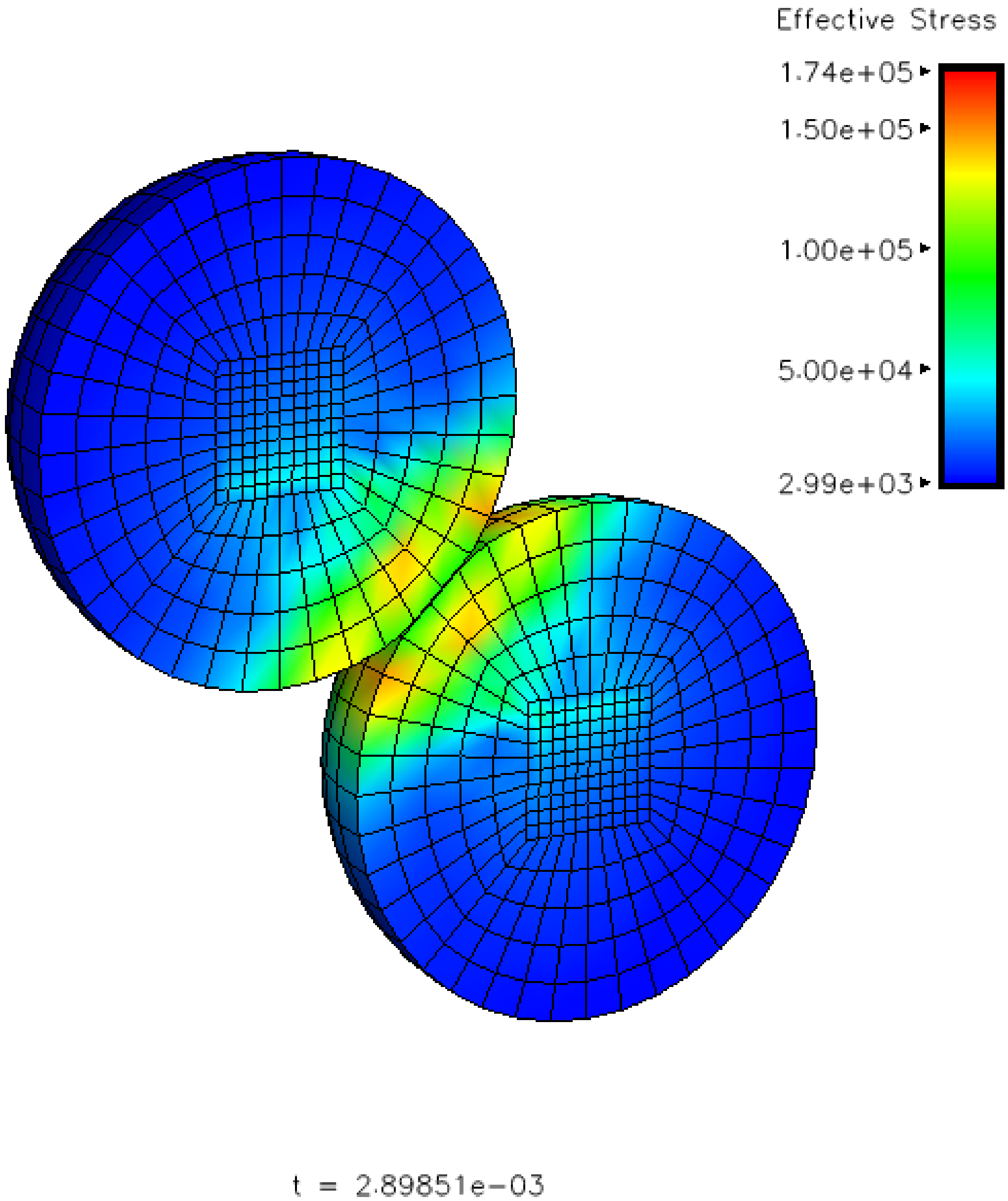
$\approx$ 1 minute for contact (5

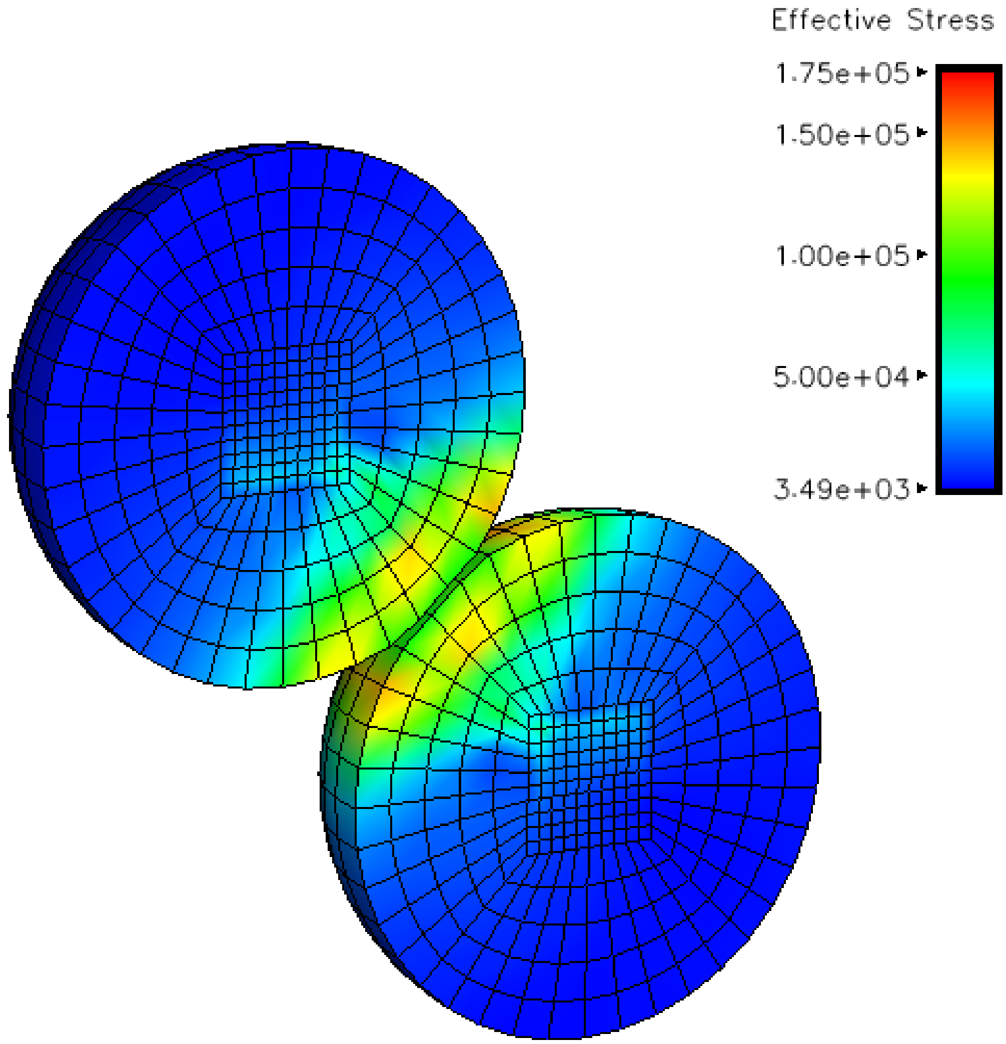

Effective Stress
0.00e+00
t = 0.00000e+00


Effective Stress
2.62e+05
2.00e+05
1.50e+05
1.00e+05
5.00e+04
7.77e+02
t = 8.99510e-04

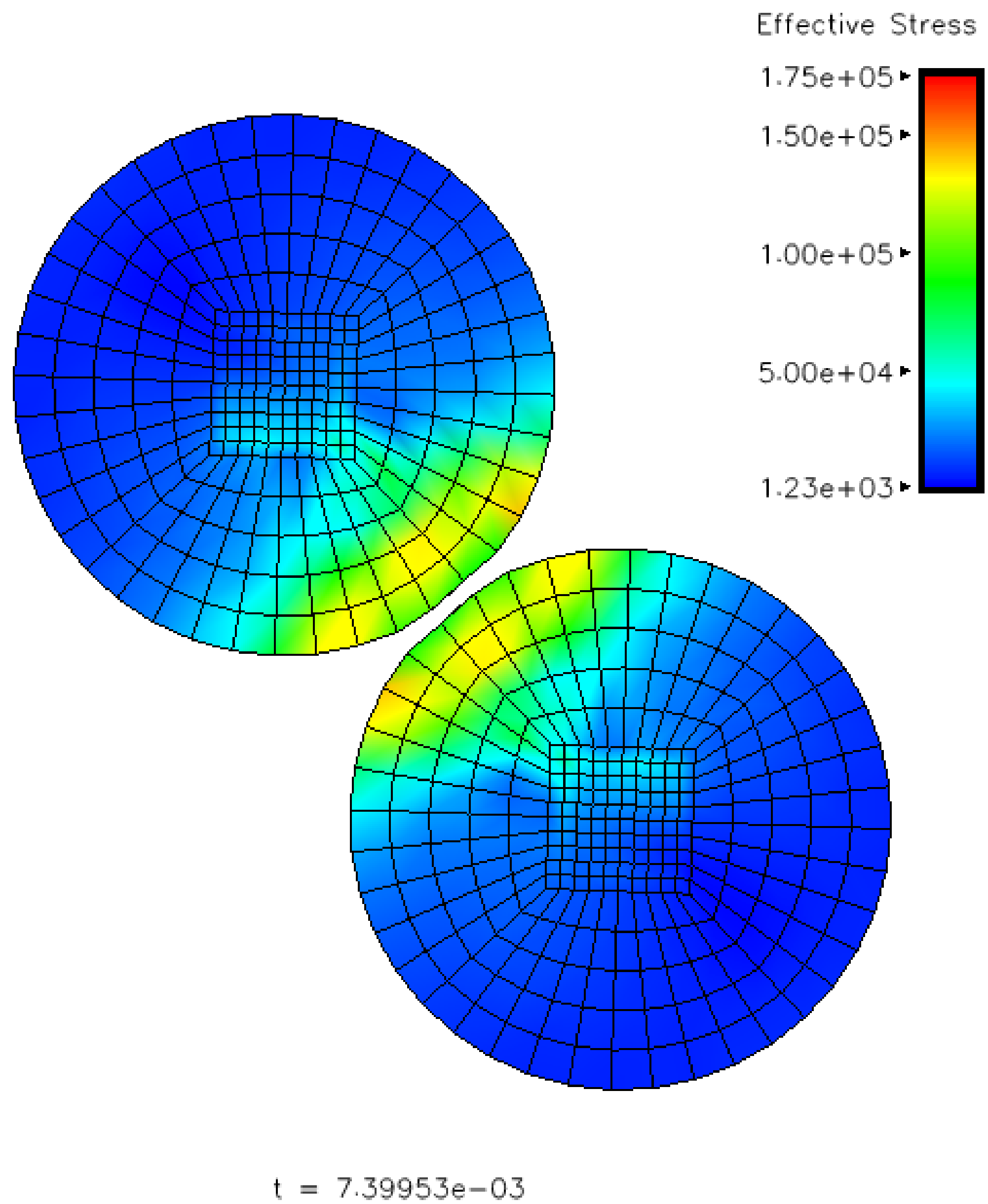

Effective Stress
2.42e+05
2.00e+05
1.50e+05
1.00e+05
5.00e+04
3.26e+03

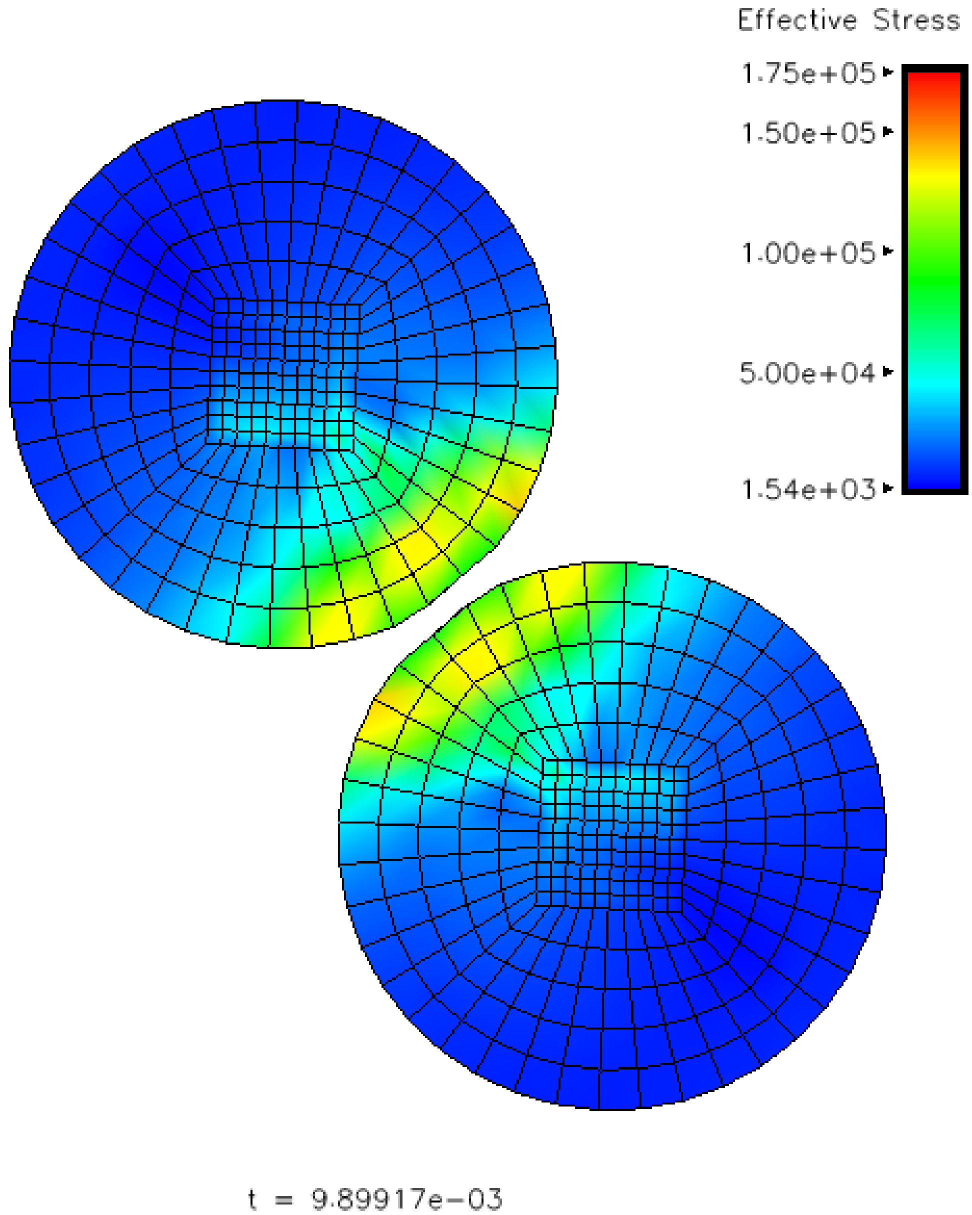
t = 1.89959e-03


Effective Stress
1.74e+05
1.50e+05
1.00e+05
5.00e+04
2.99e+03
t = 2.89851e-03

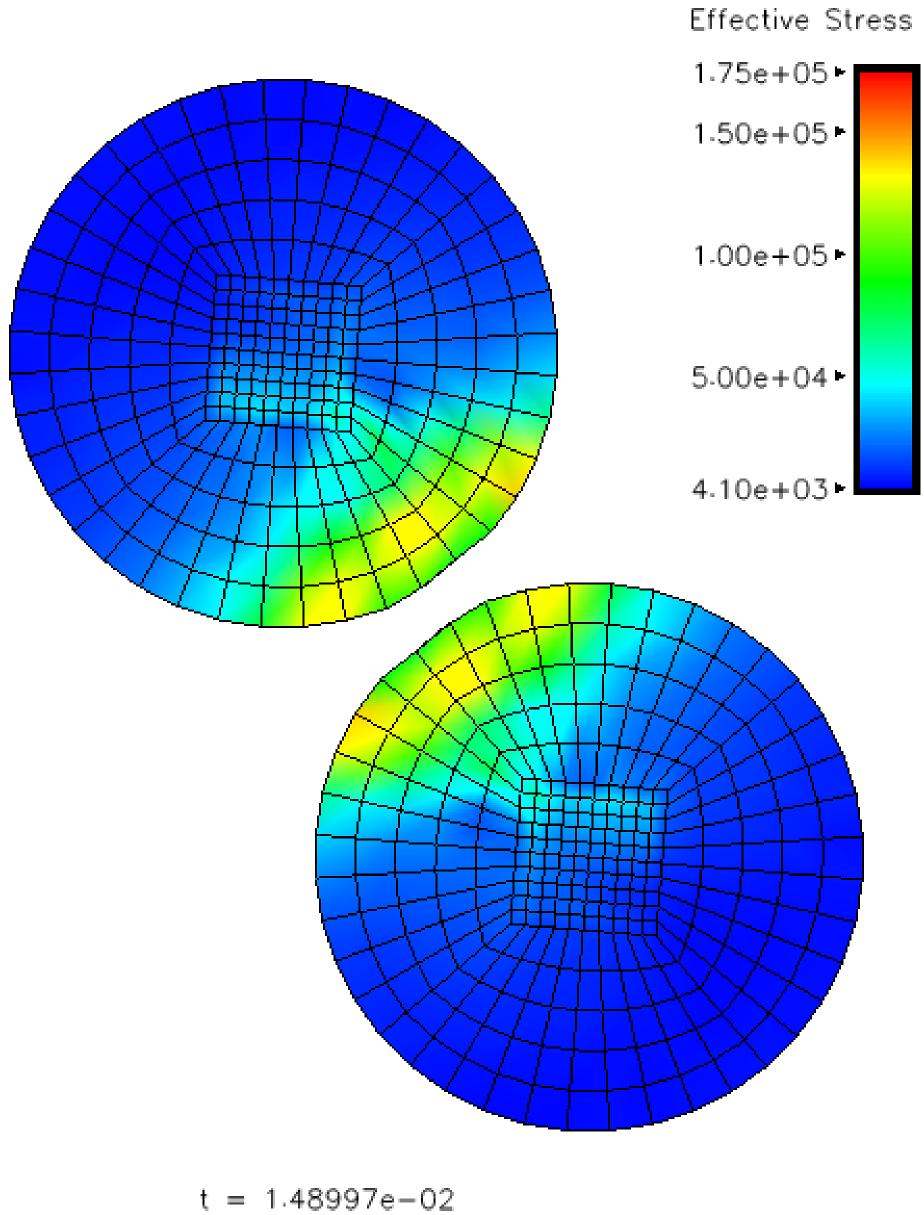


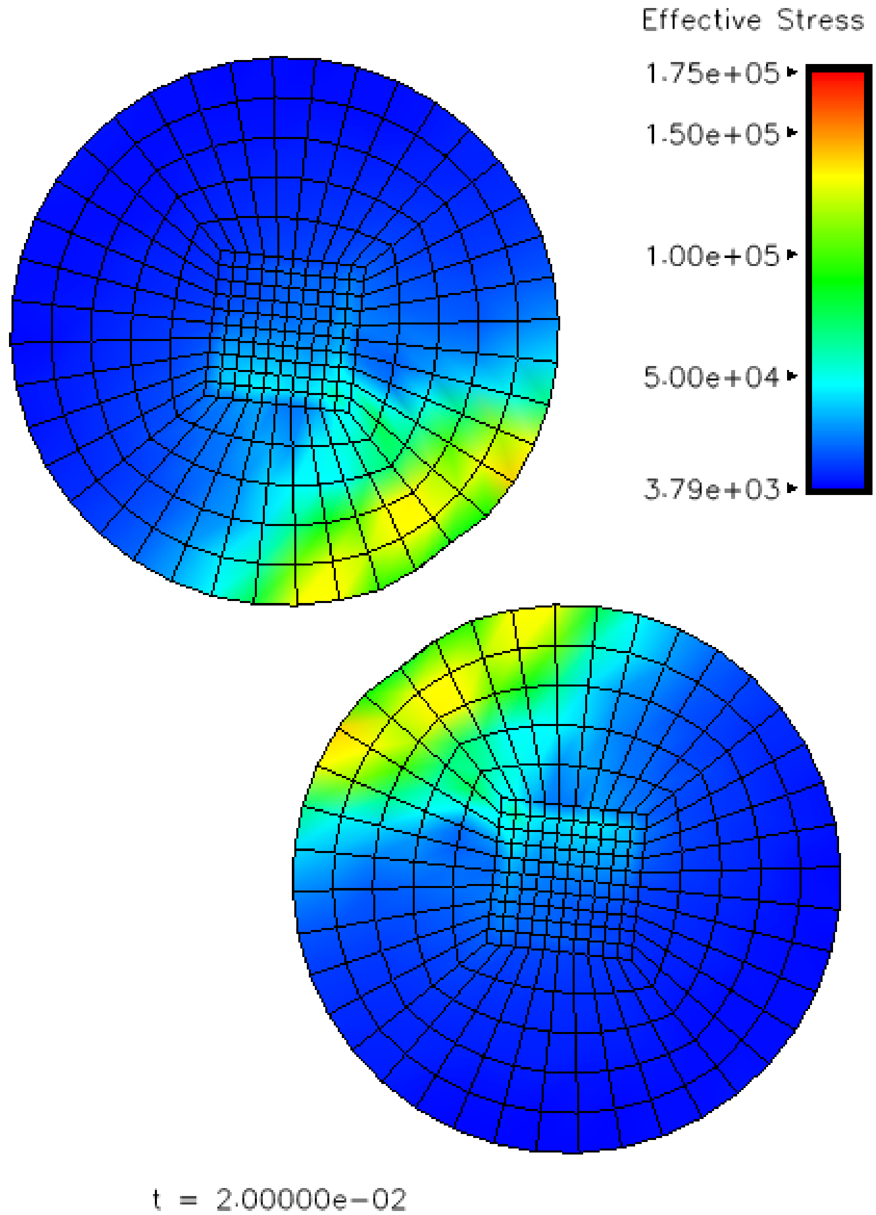

t = 4.89875e-03

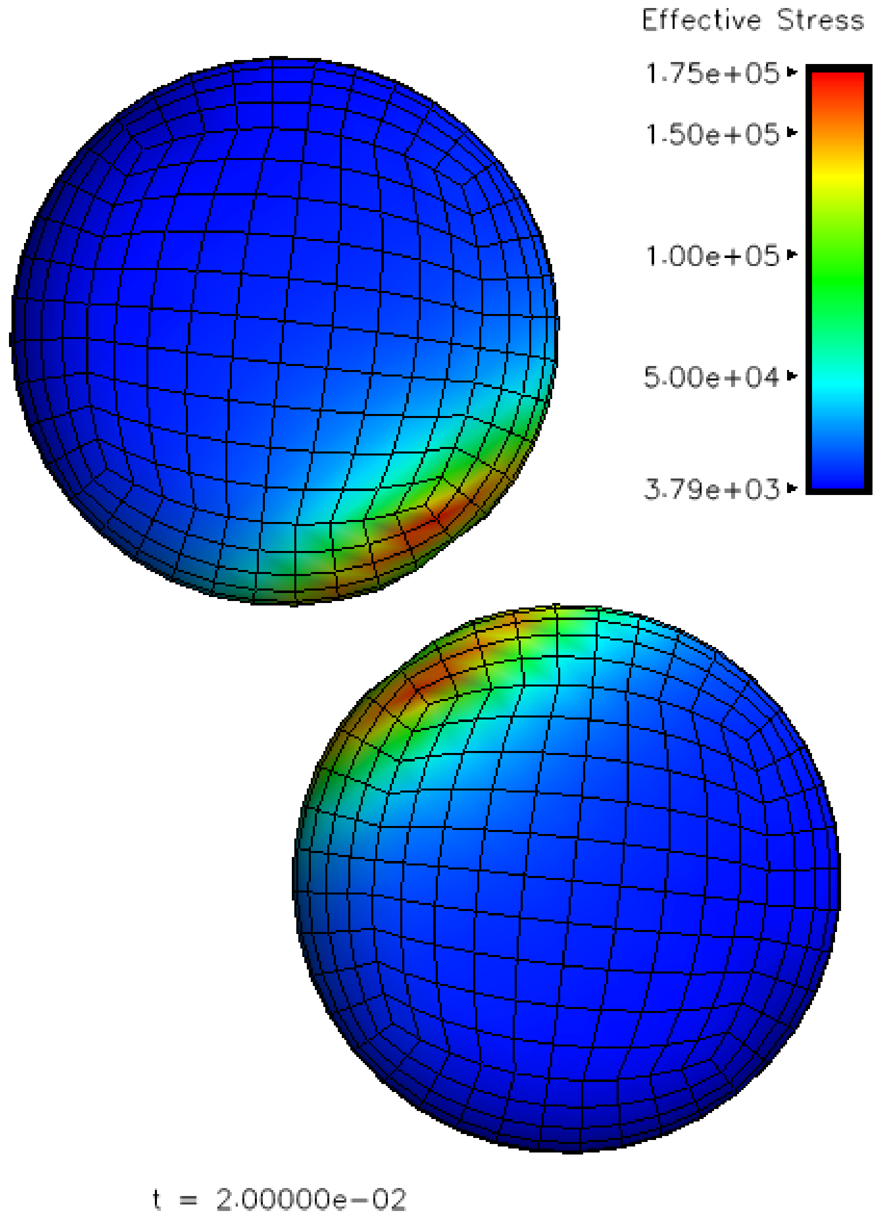

Effective Stress
1.75e+05
1.50e+05
1.00e+05
5.00e+04
1.23e+03
t = 7.39953e-03



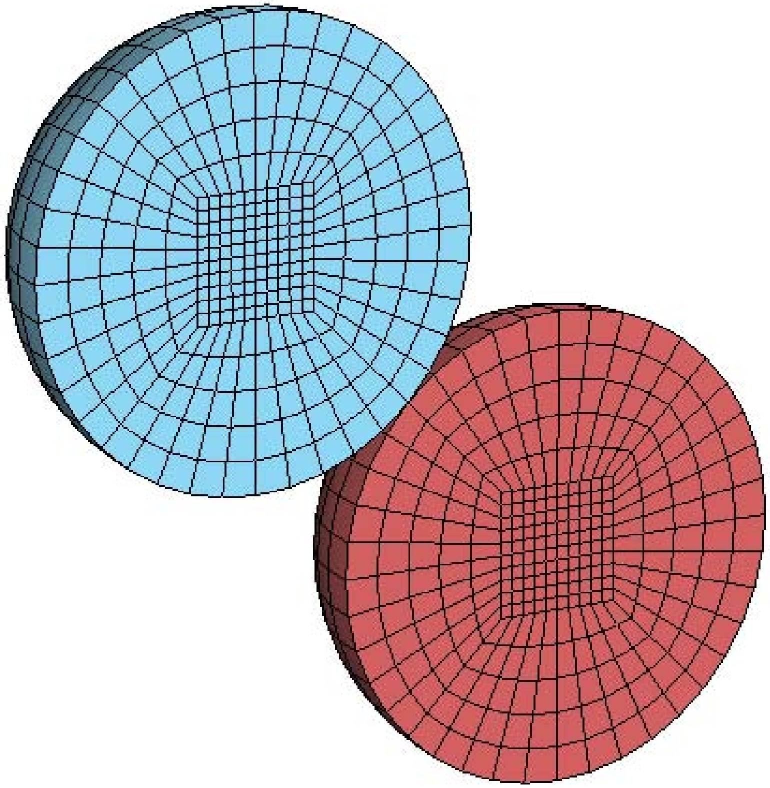
Effective Stress
1.75e+05
1.50e+05
1.00e+05
5.00e+04
1.54e+03
t = 9.89917e-03

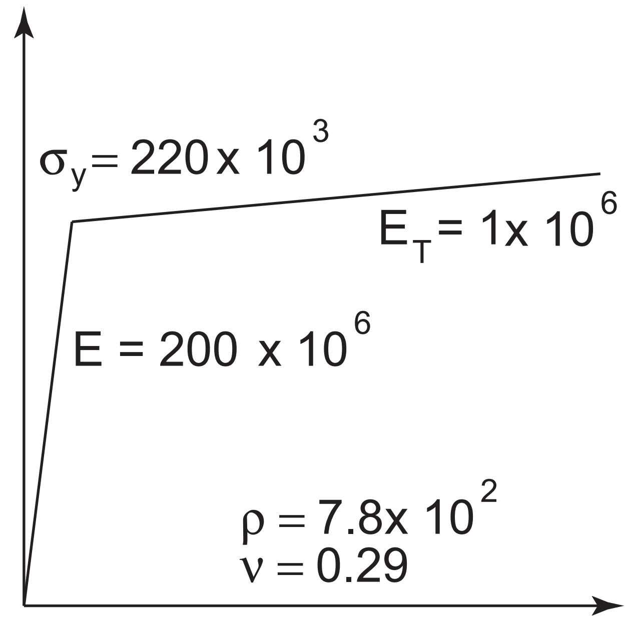

Effective Stress
1.75e+05
1.50e+05
1.00e+05
5.00e+04
4.10e+03
t = 1.48997e-02

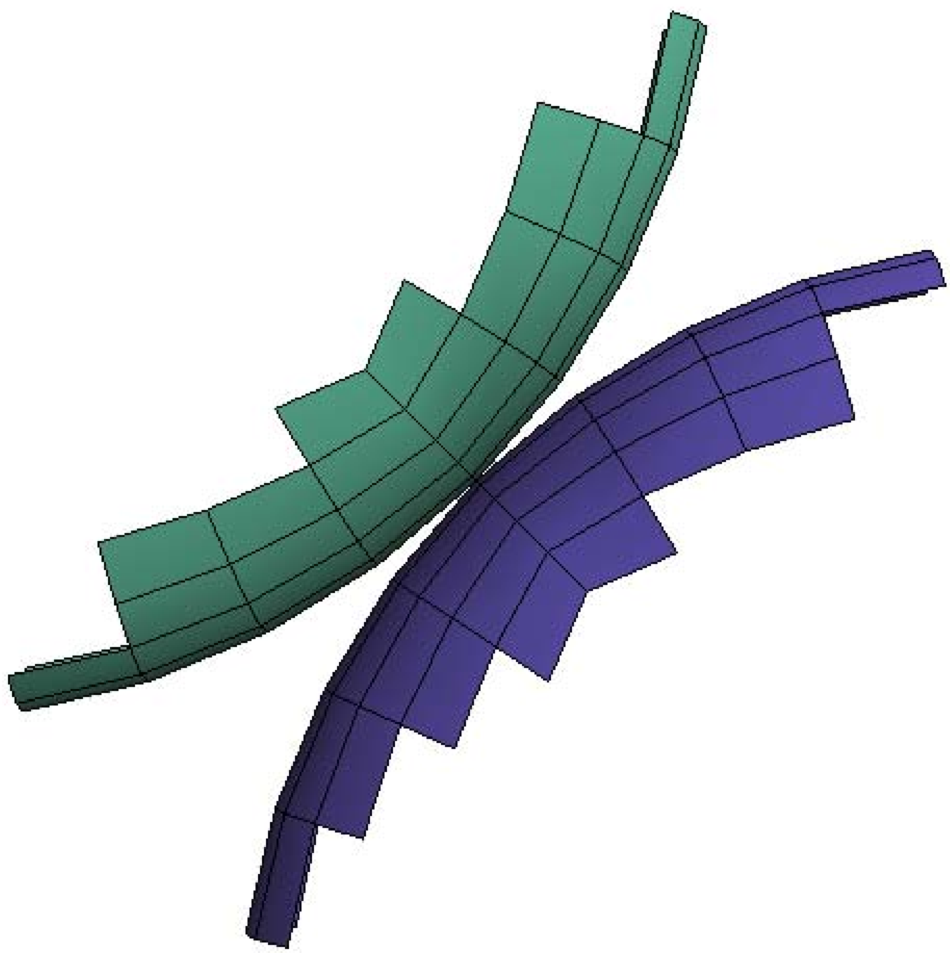

Effective Stress
1.75e+05
1.50e+05
1.00e+05
5.00e+04
3.79e+03
t = 2.00000e-02


Effective Stress
1.75e+05
1.50e+05
1.00e+05
5.00e+04
3.79e+03
t = 2.00000e-02

# Material







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- `http://www.cs.berkeley.edu/~flab/elas/elasticity.html`