A New Parametrization for the Scale Dependent Growth Function in General Relativity

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Outline

1. **Dangers of the growth equation in the Newtonian gauge**
   - Deriving the growth equation
   - Testing the growth equation
   - An improved growth equation
   - Gauge issues
2. **A new parametrization for growth**
   - The standard parametrization
   - An improved parameterization
   - Testing the new parameterization
3. **Conclusions**

The growth equation

- The growth equation is written as:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho\delta = 0\,.$$

- Simple equation describing growth of perturbations.
- Basis for parameterizations describing growth of structure[1].
- Can be used to distinguish GR and MG[2].

[1] Linder (2005); Linder and Cahn (2007); Polarski and Gannouji (2007); Gong (2008)

[2] Acquaviva,Das,Hajian and Spergel (2008)

Exact (linearized) equation for growth

- Growth of linear perturbations in a generic cosmic fluid in the Newtonian gauge, *sans any simplifying approximations*[3]:

$$\ddot{\delta} + \dot{\delta}(2-3w)H + \frac{k^2}{a}w\delta + \frac{k^2}{a^2}(1+w)\Phi = 3(1+w)[\ddot{\Phi} + \dot{\Phi}(2-3w)H] + 3\dot{w}\dot{\Phi} -3H\dot{\Theta} + [-\frac{k^2}{a^2} + \frac{3H^2}{2}(1+9w)]\Theta$$

where Θ is given by

$$(\delta p/\delta\rho - w)\delta$$

[3] Dent, Dutta and Weiler (2008)

Assumptions behind the growth equation

- Assume slowly varying w. This leads to

$$\Theta = 0, \dot{\Theta} = 0$$

- Assume matter domination.

$$w \approx 0, \dot{\Phi} \approx 0, \ddot{\Phi} \approx 0$$

- Assume that the Poisson equation is true.

$$-4\pi G\delta\rho = \frac{k^2}{a^2}\Phi$$

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Growth of perturbations in a ΛCDM background

- Background:

$$\begin{aligned} 2\dot{H} + 3H^2 &= 8\pi G\rho_\Lambda \\ \dot{\rho} &= -3H\rho \end{aligned}$$

- Perturbations:

$$\begin{aligned} \ddot{\Phi} &= -4H\dot{\Phi} - 8\pi G\rho_\Lambda\Phi \\ \dot{\delta} &= 3\dot{\Phi} + \frac{k^2}{a^2}v_f \\ \dot{v}_f &= -\Phi \end{aligned}$$

- Constraints:

$$\begin{aligned} 3H\left(H\Phi + \dot{\Phi}\right) + \frac{k^2}{a^2}\Phi &= -4\pi G\delta\rho \\ \left(H\Phi + \dot{\Phi}\right) &= -4\pi G\rho v_f \end{aligned}$$

Comparison to numerics

- $\delta \rightarrow$ growth from the numerical integration
- $\delta_g \rightarrow$ growth from the growth equation
- $\Delta \rightarrow$ comparison variable

$$\Delta \equiv \frac{(\delta_g - \delta)}{\delta}.$$

Error for a given scale as function of redshift

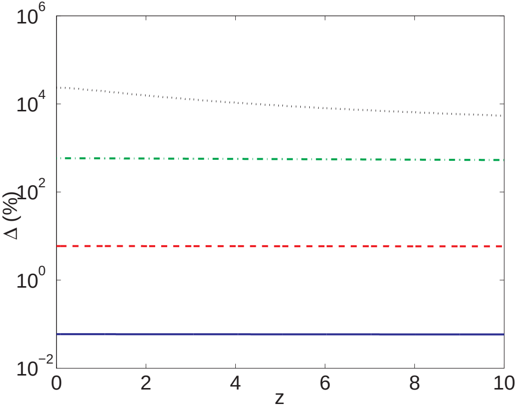


Figure: The (percentage) error Δ in the growth (δ_g) predicted by the usual growth equation as a function of redshift, for four different scales. From the top down, the curves are for $k = .001$h Mpc^{-1}, $k = .01$h Mpc^{-1}, $k = .1$h Mpc^{-1} and $k =$h Mpc^{-1} respectively.

Error for a given redshift as function of scale

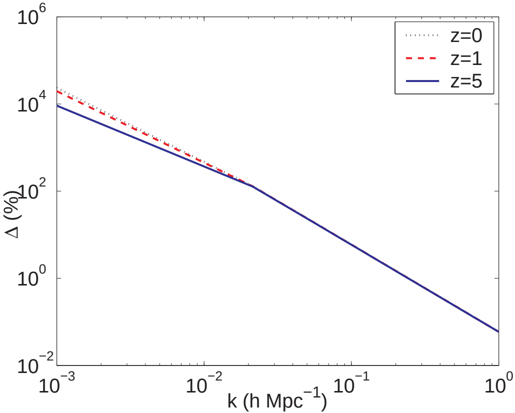


Figure: The (percentage) error Δ in the growth (δ_g) predicted by the usual growth equation as a function of scale, for three different redshifts

The central problem

In the Newtonian gauge, the growth equation is not reliable on scales larger than about $50h^{-1}$ Mpc.

The Culprit: Relativistic corrections to the Poisson equation

- The Poisson equation actually comes from the momentum constraint equation:

$$3H^2\Phi + \frac{k^2}{a^2}\Phi + 3H\dot{\Phi} = -4\pi G\delta\rho$$

- The first two terms on the LHS become comparable on large (subhorizon) scales.
- The last term on the LHS can be neglected under the assumption of matter domination.

A possible remedy

- Ignore the $3H\dot{\Phi}$ term, but incorporate the $3H^2\Phi$ term.
- This leads to an improved growth equation[4]:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{4\pi G\rho}{1+\xi}\delta = 0$$

where $\xi \equiv 3a^2H^2/k^2$

- Manifestly scale dependent, and far more accurate on large scales.

[4] Dent and Dutta (2008)

Error of the improved growth equation as function of redshift

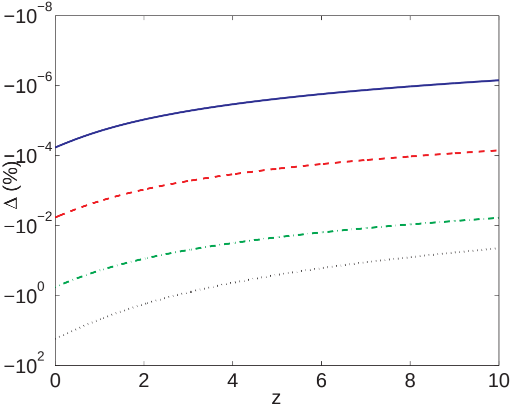


Figure: The (percentage) error Δ in the growth predicted by the improved growth equation as a function of redshift, for four different scales. From the bottom up, the curves are for $k = .001$h Mpc^{-1}, $k = .01$h Mpc^{-1}, $k = .1$h Mpc^{-1} and $k = h$ Mpc^{-1} respectively.

Error of the improved growth equation as function of scale

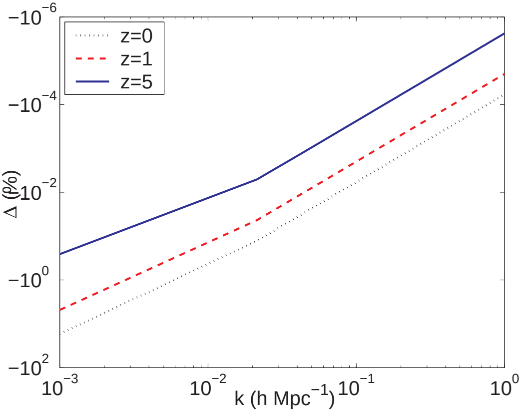


Figure: The (percentage) error Δ in the growth predicted by the improved growth equation as a function of scale, for three different redshifts.

Growth equation in synchronous and Newtonian gauges

- Well-known that the growth equation is exact and scale-independent in the synchronous gauge.
- Can be proved that the deviation of the growth equation from the true growth is exactly the deviation between $\delta_{\rm syn}$ and $\delta_{\rm cN}$[5]
- On small scales both gauges give the same answer.
- On large scales, important to consider which of these gauges is best suited to describe real astronomical observations.

[5] See Dent and Dutta (2008) for details

The growth index γ

- Growth index γ parameterizes the growth function f defined as:

$$f \equiv \frac{d \ln \delta}{d \ln a} = \Omega_m(a)^\gamma$$

- Provides an excellent fit to the evolution of $\delta(a)$ according to the growth equation.
- For dark energy models in a flat Universe with slowly varying equation of state $w \approx w_0$, the growth is well approximated by the choice

$$\gamma = \frac{3\,(w_0 - 1)}{6w_0 - 5}$$

- Current constraints on γ fairly weak: $\gamma = 0.674^{+0.195}_{-0.169}$. Expected to tighten with more detailed weak lensing surveys.

Scale-dependent parametrization

- As shown earlier, the growth equation is not reliable on scales larger than around $50h^{-1}$ Mpc - hence a new, scale dependent parameterization is necessary.
- Based on the improved growth equation, we propose the following scale dependent parameterization[6]:

$$\frac{d\ln\delta}{d\ln a} = \frac{\Omega_m(a)^\gamma}{1 + \frac{3H_0^2\Omega_{m0}}{ak^2}}$$

- Excellent agreement with the true growth upto horizon scales for all redshifts.
- Works very well with variable w (dynamical dark energy) models.

[6] Dent, Dutta and Perivolaropoulos, 2009

New vs Standard parametrization

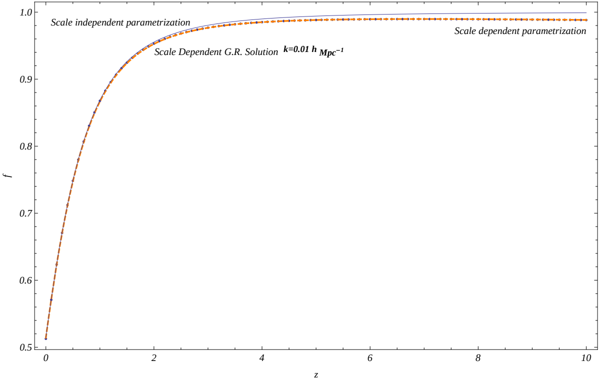


Figure: The growth factor f obtained from the solution of the general relativistic system ($k = 0.01 h^{-1} Mpc$, $\Omega_{0\mathrm{m}} = 0.3$, ΛCDM, dotted line) compared with the scale independent parametrization (continuous line) and the corresponding generalized scale dependent parametrization (thick dashed line).

New vs Standard parametrization

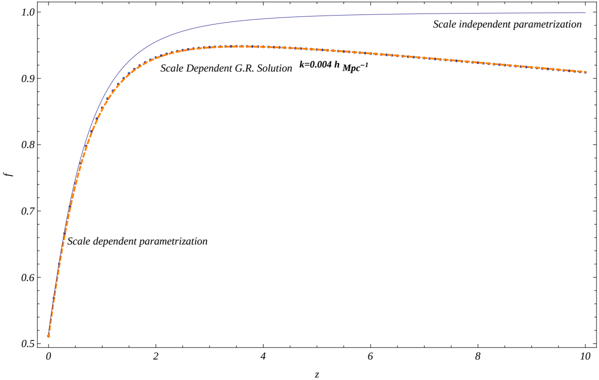
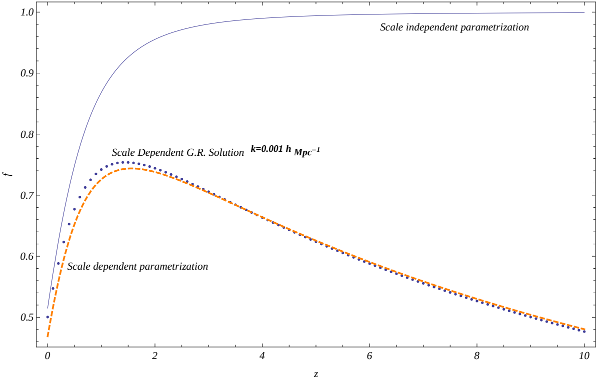


Figure: The growth factor f obtained from the solution of the general relativistic system ($k = 0.004 h^{-1} Mpc$, $\Omega_{0\mathrm{m}} = 0.3$, ΛCDM, dotted line) compared with the scale independent parametrization (continuous line) and the corresponding generalized scale dependent parametrization (thick dashed line).

New vs Standard parametrization



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Summary and conclusions

- Demonstrated that in the Newtonian gauge, the growth equation can be surprisingly inaccurate on surprisingly small (sub-horizon) scales (100% disagreement at about $50h^{-1}$ Mpc) as a result of GR corrections to the Poisson equation.
- The ordinary growth equation is therefore unsuitable as a basis for parameterizations of the growth function (to distinguish GR and MG etc) especially on large scales.
- Proposed an improved growth equation which accounts for GR corrections to the Poisson equation.
- Based on the improved growth equation, proposed a new scale-dependent parameterization for the growth function which accurately reproduces the true growth upto horizon scales.