Evaluating hysteresis in earth materials under dynamic resonance

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Abstract. A lumped parameter model is derived for studying hysteretic effects in resonant bar experiments on rock. The model uses equations of state obtained by approximating closed hysteresis loops in the stress-strain plane by parallelograms. The associated approximate nonlinear state relations have a sound speed (modulus) that takes two values. Assuming hysteresis and discrete memory to be the primary nonlinear mechanisms, periodic solutions corresponding to these equations of state are obtained analytically for single-frequency continuous wave drivers, and their frequency spectral densities are analyzed. In this simple approximation, if hysteretic contributions to the signal speed are a correction to the linear elastic signal speed (i.e., the parallelogram is narrow), the model predicts that the spectral density at even multiples of the source frequency is zero, and an approximate "pairing" of amplitudes is predicted for odd harmonic multiples. Comparison of the model spectrum with experimental data shows the model to be qualitatively correct. We conclude that hysteresis is an important mechanism in rocks. We consider the model to be a prototype.

Introduction

Nonlinear elastic wave propagation experiments have been conducted at Los Alamos National Laboratory and the Institut Français du Pétrole as part of an effort to determine the nonlinear state parameters in earth materials [e.g., Meegan et al., 1993; Johnson et al., 1996; B. Zinszner et al., Influence of change in physical state on elastic nonlinear response in rock: Effects of effective pressure and water saturation, submitted to Journal of Geophysical Research, 1996]. Three types of experiments are being conducted with this goal in mind: static stress-strain, dynamic pulse propagation, and dynamic resonance experiments [see, e.g., Johnson and Rasolofosaon, 1996]. The dynamic experimental evidence suggests that more familiar approaches to modeling a nonlinear resonant system, such as the Duffing equation [see Stoker, 1950], do not predict the empirical behavior of the harmonic spectrum and resonant frequency alteration for earth materials [Guyer et al., 1995a, b; Johnson et al., 1996]. That is, classical predictions of harmonic distribution and resonant frequency shift as a function of drive amplitude in resonance experiments on earth materials do not match observation. This same conclusion has also been obtained from pulse mode wave experiments [Ten Cate et al., 1996; Van Den Abeele and Johnson, 1996; Kadish, 1995; Kadish et al., 1996].

It is clear from static tests on earth materials that hysteresis and discrete memory are characteristic [Boitnott, 1993; Holcomb, 1978, 1981; Gardner et al., 1965; Birch, 1966]. These two characteristics, together with the poor predictions obtained from classical nonlinear wave theory [Guyer et al., 1995a, b] suggest that hysteresis and discrete memory may play an important role in propagating and standing waves. There is considerable experimental evidence suggesting that hysteresis and discrete memory and/or other nonlinear phenomena strongly effect propagating and standing waves at strain levels as low as 10^-7 under ambient conditions [Ostrovsky, 1991; Guyer et al., 1995a; Johnson et al., 1996].

We introduce a simple lumped parameter model to evaluate the effects of hysteresis in resonant bar experiments. It will be demonstrated that the lumped parameter model (also "zero dimensional") predicts resonant periodic motions whose spectral amplitudes exhibit qualitative properties similar to those observed in experiments. The term hysteresis is usually employed in two contexts: history-dependent multivalued state relations in nonlinear materials (e.g., between stress and strain in rocks and between the magnetic field and magnetic induction in magnetic materials) and the discontinuous relations observed between input and output experimental parameters (e.g., frequency versus amplitude) in driven nonlinear systems. Both of these hysteretic phenomena are observed in rocks. Because the latter form is common to almost all nonlinear systems (e.g., Duffing's equation), its presence cannot be used to deduce the existence of a multivalued state relation. The lumped parameter model described in this paper uses a multivalued stress-strain relation that is amenable to mathematical analysis of oscillations driven in resonant bar experiments. Although the modeling procedure accommodates other nonlinearities, the multivalued nature of the state relation is the only nonlinearity considered here in order to facilitate analysis. (It is also the only source of dissipation considered.) Consequently, one does not expect complete quantitative agreement with data from nonlinear resonant bar experiments. However,
Figure 1. The model hysteresis loop used in this paper, for the special case where \( \sigma_0 = 0 \) (parallelogram), and a more physically realistic version (bold line) in stress-strain \((\sigma-e)\) space. For the approximating parallelogram, there are precisely two signal speeds (equivalently elastic moduli). The arrowheads indicate direction of increasing time.

the model does allow a qualitative determination of the relative strengths of hysteresis vis-à-vis other nonlinear effects. In particular, the hysteretic spectral signature exhibits unique and easily identified properties for weak nonlinearity.

In the following section an overview of the theory is provided, but the details have been placed in the appendices. Spectral data from recent resonant bar experiments are presented in the subsequent section. A discussion of the lumped parameter model and the pertinent spectral properties it predicts for resonant bar experiments follows the data section.

Wave Equations With Hysteresis

One-dimensional nonlinear pulse propagation in ideal and nonideal elasticity is governed by coupled wave equations for the stress \( \sigma \) and particle velocity \( v \). The local signal speeds, \( c_{signal} \), are given by

\[
c_{signal} = \sqrt{\frac{d\sigma}{de}} \rho^* \quad (1)
\]

where \( e \) is the local strain and \( \rho^* \) is the mass density of the unstressed sample. In nonideal materials the stress-strain relation is not one-to-one. Hysteresis and other plasticity phenomena result in multivalued state relations (in the remainder of this paper, use of the term hysteresis will be restricted to mean phenomena associated with multivalued state relations as opposed to hysteresis of the resonance frequency-amplitude response). An example of such a relation is illustrated in Figure 1. The curvilinear loop in the figure represents an empirical path. Arrows along the path indicate direction with increasing time. The counterclockwise direction corresponds to hysteresis being dissipative. The parallelogram in the figure is a convenient approximation, which is discussed in detail below. Sufficiently simple periodic evolution of stress and strain results in repetition of closed loops of the type shown in Figure 1. As illustrated by the curvilinear loop, hysteretic state relations usually exhibit discontinuous variations in local signal speed (or modulus) when the time derivative of stress changes sign. The material responds differently to increases in stress than it does to decreases. The hysteretic response is generally attributed to the compliant features in the material such as cracks, grain-to-grain contacts, etc. The discontinuities in signal speed produce shocks not seen in the limited ideal elasticity. The shocks are space-time paths across which derivatives of stress and particle velocity are not continuous. Evidently, it is necessary to account for local histories of stress for accurate numerical simulations of pulse propagation in hysteretic materials.

The simplest mathematical models of signal speed in a hysteretic material are obtained by approximating curvilinear hysteresis loops by parallelograms, an example of which is illustrated in Figure 1. In this paper approximations of the form

\[
\frac{1}{\rho^*} \frac{d\sigma}{de} = c_0^2 (1 + \frac{\text{sgn}(\sigma - \sigma_0)^2}{h_+}) \quad (2)
\]

are used. Here, \( c_0^2 \) is the square of the constant "elastic" signal speed of the medium, and the \( h \) are hysteresis constants (subscripts refer to the sign of the sgn function).

Effectively, the medium behaves like two different elastic materials. The sgn function and the \( h_+ \) provide a mathematical prescription for switching between one material and the other. The signs of the hysteresis constants determine dissipative properties of hysteresis. If they are negative, as will be assumed in this paper, hysteresis is a dissipative mechanism. It is the only dissipative mechanism and nonlinear effect included in this treatment. The sgn function of the time derivative of \((\sigma - \sigma_0)^2\) determines the times at which the change in elastic behavior occurs. The values of the hysteresis constants \( h_+ \) together with the elastic signal speed determine the slopes (i.e., moduli) of the stress strain relation on the parallelogram approximation to the hysteresis loop as shown in Figure 1 for the special case \( \sigma_0 = 0 \). During propagation, the speeds are piecewise constant in time, greatly simplifying analysis.

Following the arrowheads on the curvilinear loop indicating the direction of time in Figure 1, abrupt changes in slope occur when \((\sigma - \sigma_0)^2\) is maximum, \( \sigma_{max}^2 \). These abrupt changes are physical. (When the maximum is achieved, the time derivative of \((\sigma - \sigma_0)^2\) changes sign and the material switches elastic states.) Figure 1 also shows that the slopes at common values of \( \sigma \) depend on whether \((\sigma - \sigma_0)^2\) is increasing or decreasing with time.

With the parallelogram approximation to the closed loop, discontinuities in signal speed also occur when \(\sigma - \sigma_0\) changes sign and require special treatment. These discontinuities are not physical. Their presence is a price one pays for the mathematical simplification gained by introducing the parallelogram approximation to the true hysteresis loop. Equation (2) could be made more physically realistic by smoothing to provide continuous signal speeds at \( \sigma = \sigma_m \) as illustrated by the curvilinear path in Figure 1; however, the analysis is much more difficult and, we believe, would not add significantly to the physics contained in the model. Equation (2) may be regarded as a limiting case of expressions not having this discontinuity. In order to avoid nonphysical shocks when using (2), continuity of first derivatives of stress and particle velocity will be imposed when \( \sigma - \sigma_0 \) changes sign.

In Appendix A, a lumped parameter model for temporal oscillations driven in resonant bar experiments is derived from propagation equations with signal speeds modeled using (2). The nonlinear oscillator equations for the evolution of the dimensionless scaled lumped stress \( s(t) \) and particle velocity \( w(t) \) are (see Appendix A)
Figure 2. (top) Single-frequency source spectrum, and (bottom) detected spectrum of the type predicted.

\[
\frac{d^2}{dt^2} \left( s^+(t) \right) + (1 \pm \delta)^2 \Omega^2 \left( s^\mp(t) \right) = -\left( \frac{dv(0, t)}{dt} \right) \left( \frac{d^2v(0, t)}{dt^2} \right)
\]  

where \( v(0, t) \) is the particle velocity applied at the end of the bar at \( z = 0 \). Here we note that both \( \delta \) and \( \Omega \) are positive constants that are properties of the material. \( \Omega \) is an angular frequency, and \( \delta \) is a hysteretic strength parameter. The plus and minus signs appearing in front of \( \delta \) and as superscripts of \( s(t) \) and \( w(t) \) are those of the \( \text{sgn} \) function in (2).

In the absence of hysteresis, \( \delta = 0 \), and application of a source function \( v(0, t) \) having angular frequency \( \Omega \) produces unbounded stresses and particle velocities. For \( 0 < \delta < 1 \) the model of the hysteresis effect is dissipative, and solutions to (3) are bounded. In Appendix B, (3) is solved for \( \delta << 1 \) (weak hysteresis) when \( v(0, t) \) is periodic with angular frequency \( \Omega \). In Appendix C the acceleration frequency spectrum of those solutions is analyzed. Characteristic properties of solutions in resonance with the driver are that to order 1 in the asymptotic expansion in small \( \delta \), (1) terms corresponding to even multiples of \( \Omega \) are absent, i.e., have zero amplitude (see equation (C3)), and (2) there is an approximate pairing of the acceleration amplitudes of terms corresponding to higher odd multiples of \( \Omega \) (i.e., ratios of the amplitudes of the 5\( \Omega \) and 7\( \Omega \) terms are equal to 1 within 2%, while those of the 9\( \Omega \) and 11\( \Omega \) terms, etc., are equal to 1 within \(< 1\% \) (see equations (C5) and (C6)).

These results are illustrated qualitatively in Figure 2. The top portion of the figure shows the spectrum of the driver. The bottom portion of the figure shows a detected spectrum of the type predicted by the model.

**Comparison With Experimental Data**

A schematic of a resonant bar experiment is shown in Figure 3a along with the typical character of nonlinear response in resonance in rock (Figure 3b). The bar, occupying \( 0 < z < L \), is driven at \( z = 0 \) with a periodic velocity \( v(0, t) \) while the end at \( z = L \) is stress free, \( \sigma(L, t) = 0 \). The resonance peak shifts downward with increasing drive level as a result of the nonlinear response of the material. The plot also illustrates the hysteretic nature of the frequency-amplitude response of a resonant bar driven at nonlinear levels; the upgoing and downgoing frequency curves are very different from each other when the bar is driven at nonlinear levels.

Figure 4 illustrates experimental harmonic measurements from three rock types: Fontainebleau sandstone, Lavoux limestone, and Meule sandstone [see, e.g., Lucet and Zinszner, 1992]. The detected acceleration is plotted on the horizontal...
For analytical purposes, the simplest closed hysteresis loop in elastic materials whose state relation is hysteretic. Periodic solutions were obtained analytically, and their spectra were compared with the experimental data.

The lumped parameter models were applied to materials driven in resonance from one boundary and free at the other. This configuration is the one used in nonlinear resonant bar experiments. Periodic solutions driven by a single-frequency source were obtained for approximating stress-strain loops. For analytical purposes, the simplest closed hysteresis loop in stress-strain coordinates is a parallelogram because it corresponds to a stress-strain relation with only two signal speeds. For weak hysteresis, the theory predicts the amplitude of resonant motions to scale inversely with the strength of the hysteresis. (This is because hysteresis was the only damping mechanism included in this treatment; see equation (C3)). In addition, at odd multiples of the source frequency, the spectra of these solutions exhibit pairing: the ratios of the amplitudes of the seventh to fifth, eleventh to ninth, etc., are very close to unity. Because the only nonlinearity present is hysteresis, even harmonics have zero amplitude in this approximation. If other nonlinearities were included in the model, e.g., nonconstant elastic signal speed, even harmonics would be present and the pairing of odd harmonic amplitudes would be weakened.

The pairing of odd harmonic amplitudes predicted by the theory was qualitatively confirmed by experiment; i.e., linear extrapolation of the \( \omega_7/\omega_5 \) ratio to low-amplitude drive was consistent with theory. (Note that higher frequencies could not be used because these data were beneath the noise floor.) The linear extrapolation to low amplitude is not conclusive; however, as is shown in Figure 5, the extrapolations indicate an
Figure 5. Ratio of measured harmonic amplitudes $\omega_7/\omega_5$ versus detected acceleration for (a) Meule sandstone, (b) Lavoux limestone, and (c) Fontainebleau sandstone. Linear fit shown and intercept of fit noted on plots.
increasing ratio as amplitude decreases. In the absence of hysteresis, ratios of spectral amplitudes would be expected to approach zero with decreasing amplitude on the basis of classical perturbation analysis [Stoker, 1950]. Even with hysteresis, at low drive levels the absolute harmonic amplitudes approach zero, but their ratio, according to this model, does not. Consequently, the extrapolations of the observations are consistent with hysteresis being a dominant mechanism and suggest that care should be taken when applying classical perturbation methods to hysteretic materials.

The lumped parameter model derived here can be used with more complete state relations. The only dissipative and non-linear mechanism considered was that due to hysteresis. The model can accommodate other damping mechanisms and non-linearities. Further development of the model will seek a more complete account of observed spectral properties by expanding the prototype to include additional phenomena.

Appendix A: Lumped Parameter Model

The simplest one-dimensional model for studying compressional wave propagation in elastic and elastic-plastic media is the first-order $2 \times 2$ system consisting of the equations of continuity and force balance for the stress, $\sigma(x, t)$, and particle velocity, $v(x, t)$ (the laboratory position coordinate is $x$, and time is $t$). In Lagrangian coordinates (i.e., coordinates fixed in the material),

$$\frac{1}{\rho^*} \frac{\partial \sigma}{\partial t} + \frac{\partial v}{\partial z} = 0$$

$$\rho^* \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial z} = 0$$

(A1)

where $e(z, t)$ is the strain; a state, or stress-strain, relation between $\sigma$ and $e$ has been assumed; and $\rho^*$ is the mass density in the absence of stress, where the laboratory coordinate $x$ of an element is a function of its initial position, or Lagrangian coordinate $z$, so that $x = x(z, t)$ (see Figure 3). The particle velocity is the partial derivative of $x$ with respect to $t$.

In this appendix and the ones that follow, the expression for $d\sigma/de$ given by equation (2) of the main text is used. In this appendix a lumped parameter model for resonant bar experiments is derived using (A1). The method, described below, is equivalent to a spatial averaging of (A1).

The lumped parameters for the particle velocity and stress for a bar as illustrated in Figure 3a will be denoted by $v(t)$ and $\sigma(t)$, respectively. It will be assumed that a driving particle velocity, $v(0, t)$ is applied at one end of the bar, $z = 0$, and that the other end of the bar, $z = L > 0$, is stress-free. We reduce the dimensionality of the system given in (A1) by replacing spatial derivatives by difference quotients using the gradient scale lengths $Z_v$ and $Z_\sigma$ defined by

$$\frac{\partial v(z, t)}{\partial z} = \frac{v(z, t) - v(0, t)}{Z_v}$$

(A2)

$$\frac{\partial \sigma(z, t)}{\partial z} = \frac{\sigma(L, t) - \sigma(t)}{Z_\sigma} = -\frac{\sigma^2(t)}{Z_\sigma},$$

respectively. The superscripts refer to the sign of the sgn function in (2). The time derivatives of (A1) are replaced by time derivatives at values of stress and particle velocity at values between the lumped parameter and known boundary values.
\[
\frac{\partial}{\partial t} v(z, t) = \frac{d}{dt} \left[ \theta^+_v v(0, t) + (1 - \theta^+_v) v^+(t) \right]
\]

\[
\frac{\partial}{\partial t} \sigma(z, t) = \frac{d}{dt} \left[ \theta^+_\sigma \sigma(t) + (1 - \theta^+_\sigma) \sigma^+(L, t) \right] = \frac{d}{dt} \left[ \theta^+_\sigma \sigma(t) \right].
\]

The \( \theta \) are intermediate value parameters (i.e., if the \( \theta \) are assigned different values between zero and one, the quantities being differentiated take values between the boundary data and the lumped parameters). Substituting (A2) and (A3) in (A1) yields a coupled system of first-order ordinary differential equations for the lumped particle velocity parameter \( v(t) \) and stress parameter \( \sigma(t) \).

\[
\frac{d}{dt} \left[ \frac{\theta^+_v \sigma(t)}{\rho^*} \right] - \frac{1}{\rho^*} \left( \frac{d \sigma}{d\varepsilon} \right) \left[ \frac{v'(t) - v(0, t)}{Z_v^*} \right] = 0
\]

\[
\frac{d}{dt} \left[ \theta^+_\sigma v(0, t) + (1 - \theta^+_\sigma) v^+(t) \right] + \frac{\sigma^+(t)}{\rho^* Z_v^*} = 0
\]

Constraints exist that limit the number of independent gradient scale lengths and intermediate value parameters in the system. Because the terms that are not time derivatives are integrable when the derivative of the stress changes sign, the terms being differentiated are continuous. This implies that there are only two independent intermediate value parameters

\[
\theta^+_v = \theta^+_v = \theta^+_v \quad \theta^-_v = \theta^-_v = \theta^-_v
\]

Moreover, because the time derivative of \( \sigma(t) \) is continuous when \( \sigma(t) \) changes sign, the first relation in (A4) yields the constraint

\[
\left( \frac{1}{\rho^* Z_v^*} \frac{d \sigma}{d\varepsilon} \right) = \left( \frac{1}{\rho^* Z_v^*} \frac{d \sigma}{d\varepsilon} \right) = \frac{C^2}{Z_v^*}
\]

Equation (A6), together with the second relation in (A5) and continuity of \( v(t) \), implies that the first derivative of the \( \sigma(t) \) is a continuous function of time. It must therefore vanish at maxima and minima of \( \sigma(t) \). At stationary values of \( \sigma(t) \), the \( v(t) - v(0, t) = 0 \).

Differentiation of (A4) using (A5) and (A6) yields oscillator equations for the stress and particle velocity parameters:

\[
\frac{d^2}{d\tau^2} \left( \frac{\sigma^+(\tau)}{\rho^*} \right) + \Omega^2 \left( \frac{\sigma^+(\tau)}{\rho^*} \right) = -\frac{C^2}{\theta^+(1 - \theta^+)} Z_v^* \frac{d}{dt} v(0, t)
\]

\[
\frac{d^2}{d\tau^2} \left( v^+(\tau) - v(0, t) \right) + \Omega^2 \left( v^+(\tau) - v(0, t) \right) = -\frac{1}{(1 - \theta^+)} \frac{d^2}{dt^2} v(0, t)
\]

where

\[
\Omega^2 \equiv \frac{1}{\theta^+(1 - \theta^+)} \frac{C^2}{Z_v^*} \left( \frac{1}{Z_v^*} \right)
\]

\[
= \frac{C^2 (1 + \delta)^2}{\theta^+(1 - \theta^+)} Z_v^* \frac{1}{Z_v^*} = \Omega^2 (1 + \delta)^2
\]
In terms of the dimensionless parameters of Appendix B, (A4) becomes

$$W^\pm(T) = \frac{d}{dT} S^\pm(T) \quad (1 \pm \delta)^2 S^\pm(T) = \frac{d}{dT} W^\pm(T) \quad (C1)$$

Since the dimensionless stress and strain have period $2\pi$ in $T$, it is natural to represent them as a Fourier series in sine and cosine functions periodic in $T$ on $-\pi < T < +\pi$. Odd-numbered harmonics will not appear in the series because $S(T + \pi) = -S(T)$.

Up to and including terms of order one, the first equation of (C1) is

$$\frac{d}{dT} \left[ \frac{(1 - \delta_\pi) v^\pm(T)}{V} \right] = (1 - \delta_\pi) \cos (T + \Phi)$$

$$+ (1 \pm \delta)^2 s^\pm(T) = (1 - \delta_\pi) \cos T$$

$$+ \frac{\pi^4}{8} \sin T + \frac{1}{2} \left( T \sin T + \left( 1 - \frac{\pi^2}{4} \right) \cos T \right)$$

$$+ \frac{\pi^4}{4} \left( \pi (T \cos T + \sin T) \right) \quad (C2)$$

Since the only odd function in $T$ on $-\pi/2 < T < +\pi/2$ in (C2) is the term that varies inversely with $\delta$, the only terms of order 1 in the Fourier series for the acceleration are odd harmonics of cosine functions. Writing

$$\frac{d}{dT} \left( \frac{(1 - \delta_\pi) v^\pm(T)}{V} \right) = \frac{\pi^4}{8} \sin T + \sum_{n=0}^\infty \alpha_{2n+1} \cos (2n + 1)T \quad (C3)$$

for $T$ on $-\pi < T < +\pi$, one finds for $n > 0$,

$$\alpha_{2n+1} = \frac{1}{4} \frac{(2n + 1)^2}{[n(n + 1)]^3} + n \quad n \text{ even}$$

$$\alpha_{2n+1} = \frac{1}{4} \frac{(2n + 1)^2}{[n(n + 1)]^3} - (n + 1) \quad n \text{ odd} \quad (C4)$$

Setting $n = 2k$ for $k = 1, 2, 3, \ldots$, one finds

$$||\alpha_{2(2k+1)+1}|| - ||\alpha_{2k+1}|| = \frac{1}{2} \frac{(2k)(2k+1)^2}{(2k+2)} \approx \frac{1}{32} k^4 \quad (C5)$$

where the asymptotic expression is intended for $k \gg 1$.

Setting $n = 2k + 1$ for $k = 1, 2, 3, \ldots$, one finds that for large $k$

$$||\alpha_{2(2k+1)+1}|| - ||\alpha_{2(2k+2)+1}|| \approx \frac{1}{32} k^2 \quad (C6)$$

consequently, a "pairing" of amplitudes is obtained (see Figure 2). The pairing occurs fairly early in the series and becomes more pronounced. For example, for $n = 2, 3$ ($k = 1$ in (C4)) and $n = 4, 5$ ($k = 2$ in (C4))

$$||\alpha_{7}/\alpha_{9}|| \approx 0.98, \quad ||\alpha_{11}/\alpha_{13}|| \approx 0.996 \quad (C7)$$

while $||\alpha_{7}/\alpha_{9}|| \approx 1.68$.

For $n = 0$, one obtains

$$\alpha_1 = (1 - \delta_\pi) + \left[ 1 - \left( \frac{\pi}{4} \right)^2 \right] \quad (C8)$$

**Figure B1.** Phase space representation of the closed hysteresis loop from Figure 1 for the evolution of the dimensionless stress. Arrowheads on the curve indicate the direction of variation. The plus and minus signs in quadrants correspond to the sign of the product of stress and its derivative in the quadrant. The period is $2\pi$. Two points on the solution path lying on the same straight line though the origin are equally distant from the origin and separated in time by $\pi$. 

Equation (B5) contains six $\delta$-dependent constants, $A^\pm$, $B^\pm$, $T_0$ and $\Phi$, that must satisfy the six symmetry and continuity constraints on the interval:

$$S^-(\pi/2) = -S^+(\pi/2), \quad \frac{d}{dT} S^+(T) = \frac{d}{dT} S^-(T),$$

$$S^+(T_0) = 0, \quad \frac{d}{dT} S^+(\pi/2) = \frac{d}{dT} S^-(\pi/2) = 0 \quad (B6)$$

For $\delta \ll 1$, (B6) yields

$$T_0 = \frac{\pi}{2} - \frac{2}{\pi} \delta + O(\delta^2) \quad \Phi = \frac{1}{2} T_0 + O(\delta^2)$$

$$S^+(T) = \frac{\pi^4}{8} \sin T \frac{1}{2} \left( T - \frac{\pi}{2} \right) \sin T$$

$$+ \left[ 1 - \frac{\pi^2}{2} \left( \frac{1}{2} - T \right) \right] \cos T + O(\delta) \quad (B7)$$

$$S^-(T) = \frac{\pi^4}{8} \sin T \frac{1}{2} \left( T + \frac{\pi}{2} \right) \sin T$$

$$+ \left[ 1 - \frac{\pi^2}{2} \left( \frac{1}{2} + T \right) \right] \cos T + O(\delta)$$

The linear time dependence in the asymptotic expansions for $s(T)$ in $\delta$ is due to expansions of trigonometric functions having $\delta$-dependent arguments.

**Appendix C: Acceleration Frequency Spectrum**

The time series and spectra from resonant bar experiments are usually obtained from accelerometer measurements. In this appendix we use (A4) and the stress given by (B7) to obtain the lead terms up to order 1 in the expansion of the acceleration spectrum for weak hysteresis.
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