EVALUATION OF METHODS FOR POSSIBILISITIC NORMALIZATION OF RANDOM INTERVALS

BY

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DEDICATION

I dedicate this thesis to Marcella Jaramillo, who has sacrificed so much and in doing so has contributed immensely to this endeavor and most of all to my life. I am forever indebted to you and think the world of you.

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I would first like to thank my advisor Dr. Tim Ross for his guidance and continual support. His assistance, expertise, and instruction has truly motivated me.

Secondly, I would like to thank my committee members Dr. Cliff Joslyn, and Dr. Walter Gerstle. Their expertise and instruction has truly strengthened this work.

I would also like to thank Intelligent Systems Engineering for providing me with the opportunity to study Possibility Theory.

In the same spirit, I am thankful to Gerri Cisneros Hopkins and Sandi Moening for her friendship and continuous advise.

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ABSTRACT OF THESIS

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Abstract

Reliability assessment, system failure and system analysis can involve numerous input variables whose values occur over a range of real numbers, rather than a single number. Available knowledge of the system is often limited and the available data is usually disjoint, the output data of the system is often consistent, with some overlap. Application of existing analysis theories such as probability analysis or Bayesian analysis for the aforementioned complex systems may involve a considerable amount of subjectivity on the part of the analyst. Possibility theory is currently being developed to analyze these types of systems from a more mathematical perspective rather than a subjective one.

Possibility theory is useful when the information obtained for a system is available as intervals, and more importantly as a consonant set of intervals (i.e. nested interval set). However Joslyn [8] has shown that a consistent set (see Chapter 2) of intervals is

sufficient. The difficulties arise when the interval set is neither consonant nor consistent. The two algorithms studied in this investigation transform non-useful interval sets to a more useful form. Both algorithms are based on a mathematical approach, however they do employ a small but significant amount of analyst subjectivity.

The first of these transformation algorithms, was developed by Donald and Ross [4] to transform consistent or inconsistent but overlapping data output sets to consonant sets. The second algorithm, developed earlier by Joslyn [8, 9], transforms non-consonant data to consistent data. In both algorithms the analyst must select the normalizing interval(s) to transform the original set. In Joslyn's algorithm [8, 9], only one normalizing interval is required while the algorithm developed by Donald & Ross [4] requires a collection of normalizing intervals. Each normalizing interval or group of normalizing intervals produces a somewhat different transformed set. Since little or no prior knowledge is available about the system, the analyst's selection for the normalizing interval(s) is a subjective one. The resulting transformed set may not be the "best" transformed set (from the perspective of its likeness to the original set). Therefore, the objective of the current investigation is to eliminate this subjectivity and form an automated approach to the algorithms. A transformed set that is most "like" that of the original set is desired. Thus, the ambition of this investigation is to determine a correlation between the transformed set (that is most like that of the original set) and its normalizing interval(s). This correlation will allow this transformed set to be produced consistently without the analyst intervention.

The results of this investigation indicate that this normalizing interval should be based on its size as well as its plausibility (see Chapter 2). In the algorithm presented by Donald & Ross [4], the size also plays a role but only in the normalizing intervals that are minimal; the selection of non-minimal intervals is based on: the maximal inclusion in super-intervals, maximal plausibility and the maximum number of common elements with the entire set. The results permit the algorithms to be automated, eliminate analyst subjectivity, and uniformly produce a transformed set that is most like that of the original set.

The two algorithms (due to Joslyn and Donald) are useful in a variety of engineering applications. For instance, the Donald algorithm has been employed in a reliability study [13] involving the buckling of hollow, metallic spheres. Due to the findings of the current investigation, applications such as the buckling of hollow, metallic spheres as well as other engineering problems can be accomplished in a less subjective and more efficient manner.

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Chapter 1

Introduction

1.1 Introduction

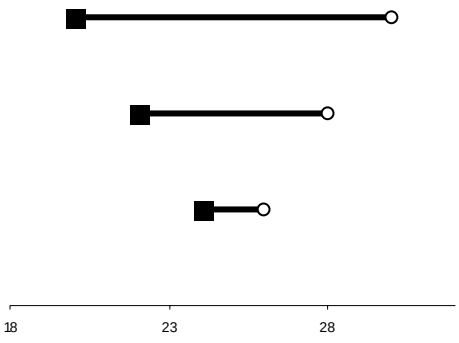
In complex systems of limited and vague prior information, such as structural system failure, the characterization and modeling of available data can be accomplished with a probabilistic analysis; however, this may require a considerable amount of subjectivity by the analyst in developing the necessary probability density functions [3]. This is especially true when the available data is not well suited for this type of modeling. According to Ross [3], due to an increase of higher consequence systems, society is no longer willing to accept an "overly optimistic safety analysis." This requires a characterization and modeling analysis method that circumvents the emphasis on analyst subjectivity and uses a more mathematical approach. Hybrid approaches are now being developed to accomplish just that; however, these hybrid approaches are susceptible to the same drawbacks of current theories used to model the available information. It is for this reason that efforts are being made to provide the analyst with a comprehensive selection of mathematical tools, in an effort to model certain systems and decrease analyst subjectivity [3].

Possibility theory finds its place within this selection of mathematical modeling tools. In

casual conversation, Possibility and Probability are often confused and used interchangeably, however there is a strong distinction between the two. A possibility is weaker than a probability, an event that is possible doesn't necessarily imply that it is probable; on the other hand an event that is probable does imply that it is also possible. Possibility theory is distinct from, but parallel to, probability theory [8]. Possibility theory is very useful in complex systems where little prior information or vague information is available for the system. Possibility theory is especially useful for modeling complex systems where the evidence is represented as a set of nested intervals rather than as a collection of distinct point values. Perhaps point values could not easily be gleaned from tests hence, intervals were reported.

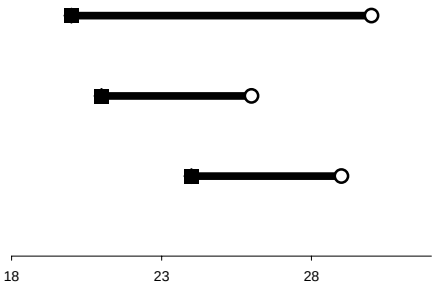
Probability theory can model such systems, but only with a significant amount of reliance on analyst subjectivity; however, possibility theory provides natural models where only interval data is available, provided certain criteria are met.

Figure 1.1a



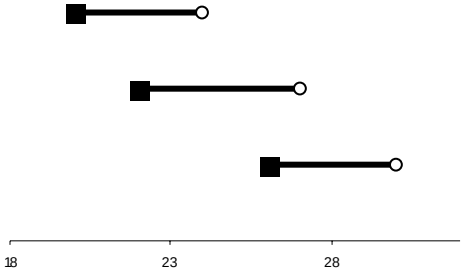
Consonant

Figure 1.1b



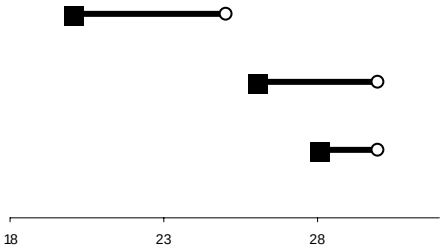
Consistent but Non-consonant

Figure 1.1c



Overlapping

Figure 1.1d



Disjoint

A consonant set of intervals (i.e. nested set of intervals) is needed to obtain a possibility measure (max value equal to 1); however, Joslyn [8] has proven that a consistent set of intervals is sufficient to obtain a normal possibility distribution and thus, a possibility measure. A consistent interval set has a common agreement among its intervals, in other words a consistent interval set has a common intersection among the intervals (see section 2.3). The difficulties arise when an interval set is neither a consonant or consistent interval set; in this case a normal possibility distribution cannot be obtained which produces anything of significance in regards to the possibility.

Donald and Ross [4] have developed an algorithm that transforms a non-nested set of interval data to a nested set, of which a normal possibility distribution is obtained. An earlier algorithm developed by Joslyn [8, 9] transforms an originally non-consistent set of data to a consistent set of data. Both algorithms employ a mathematical approach; unfortunately, both algorithms contain an empirical step that requires the analyst to make some subjective assumptions. In the consistent transformation algorithm (Joslyn [8, 9]), the analyst must select a normalizing interval from a predetermined collection of

normalizing intervals. Each normalizing interval can produce a different transformed consistent set. The same is true for the consonant transformation algorithm (Donald & Ross [4]), however a group of normalizing intervals is selected from a predetermined collection of intervals; each group producing a different transformed consonant set. The transformed set that is most "like" that of the original set is a desired outcome. The ambition of transformation is to obtain an interval set that will result in a normal possibility distribution, and that is most "like" that of the original data. In other words, in addition to obtaining a useful possibility distribution, the total information content portrayed by the transformed set should be as close to the total information content portrayed by the original set.

1.2 Objective

The objective of the current investigation is to remove analyst subjectivity from both transformation algorithms (Joslyn, Donald) and develop an automated approach to the algorithms in obtaining the transformed set that is most "like" that of the original set. This involves determining the implications that the characteristics of the normalizing interval(s) have on the transformed set. The technical approach will also involve the use of established metrics for the determination of the "likeness" and differences in uncertainty between the transformed set and the original set. The resulting values for these metrics will support the automation of both transformation algorithms and allow the removal of the analyst subjectivity. Ultimately this will consistently produce a transformed set that is most like that of the original set without the reliance on any prior

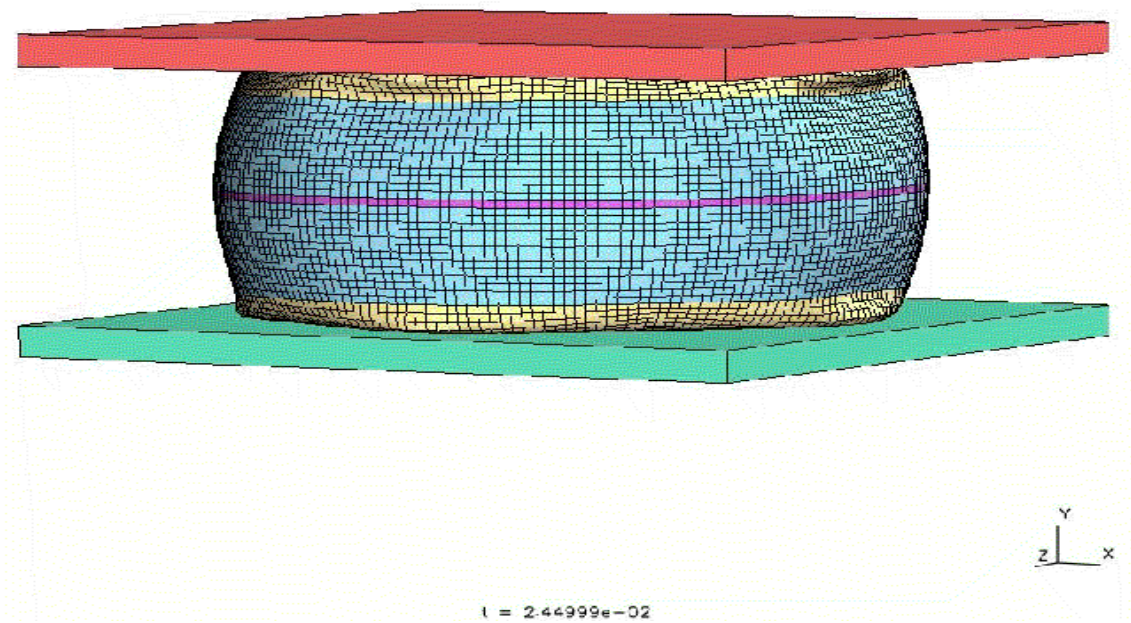
information about the system.

1.3 Utility

The two transformation algorithms are applicable to a variety of engineering problems. For example, Donald algorithm has been employed in a reliability study involving the buckling of hollow, metallic spheres (Donald & Ross [4]). To summarize [13] that application, the crushing capacity (or buckling load) of axially loaded manufactured steel marine floats is desired. Figure 1.3a and 1.3b illustrate the actual deformation and the finite element deformation analysis, respectively, for these marine floats.

Figure 1.3a

Figure 1.3b

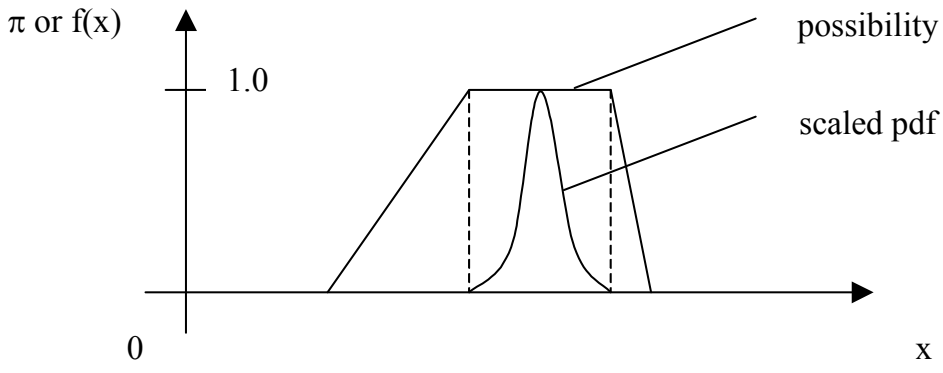


In the sphere buckling study [13] a total of 4 input parameters were investigated as variables whose values were intervals instead of single quantities, such as Young's modulus, radius of the spherical float, and the thickness of the spherical wall. Intervals for these input parameters were studied, each producing a range of outputs also expressed by an interval (i.e. crushing capacity). The collection of outputs was a non-consonant interval set, where Ross [13] applied the consonant transformation algorithm (Donald algorithm [4]) to obtain a consonant set and thus a possibility distribution for the output. Modeling this particular problem can be approached in a variety of ways, however the possibility distribution approach is less computationally expensive when compared to

other modeling approaches (i.e. Monte Carlo, mean…mean value analysis, advanced mean value analysis) [13]. Consistent transformation (Joslyn[8, 9]) is also applicable in this situation.

Possibility distributions, such as the generic one illustrated in Figure 1.3c, relate to probability distributions in the sense that a region of unit possibility spans the space of a non-zero probability distribution (e.g. a probability density function or pdf), while outside of that interval some possibility may still exist in the face of conflicting (or dissonant) evidence. As more data are acquired, the dissonance (represented by the sloping regions of the possibility distribution) diminishes and the side boundaries of the possibility distribution become steeper. One possible use of possibility distributions might be, for example, to test whether the predictive accuracy of a buckling model based on generic uncertainty data is valid for a model of a newly designed component or system.

Figure 1.3c. Relationship between probability and possibility distributions



- The approach can be used initially for all assessments to determine the regions into which more focus should be placed by subsequent probability (standard reliability) computations
- The approach can be used to quantify all variables in the simulation for which little or no data exists, or for modeling assumptions for which a probabilistic evaluation simply is not warranted since the underlying occurrence of the variable is non-random
- The approach can be used to assess those regions of the output where dissonance, or disagreement, exists in previous data or existing analytic judgments or knowledge.

Using the possibility distribution methods, as highlighted by the above findings, it is felt that the reliability assessment of structures can be streamlined in terms of cost savings and the efficient use of valuable historical data. The method also allows for the judgments and knowledge of the analyst's making the predictions more flexibility in embedding all their knowledge–not just the numeric information–into their analyses. The use of historical data to guide our analytic judgments has been used primarily in establishing a sort of classification of the appropriate methods and models to apply to any physical system.

This current investigation greatly impacts both transformation algorithms, it not only improves the uniformity but it allows the analyst to be removed from the transformation algorithms. This greatly affects the processing of possibility distributions for a variety of interval data in engineering applications such as the buckling of hollow, metallic spheres,

which can now be accomplished in a more efficient manner and by employing less subjectivity.

Chapter 2

Background

2.1 Chapter Introduction

The following chapter serves to introduce the two transformation algorithms introduced in Chapter 1, as well as other metrics used in this research. In an effort to better understand these algorithms some preliminary terms and properties are defined. Uncertainty metrics are very important in the assessment of the results; they are described in this chapter in detail. This chapter provides the insight needed for the following chapters.

2.2 Random Set

Let us first begin by describing a *random set* (S_R) for the purposes of this investigation. A random set S_R, is a random variable whose value is represented as a set-valued random variable, where there is a weight associated with each of these set-valued random variable [3]. For example, given a universe of discourse $\Omega = \{x_1, x_2, x_3\}$, then

$S_R = \{\langle\{x_1, x_2\}, 0.4\rangle, \langle\{x_2, x_3\}, 0.6\rangle\}$ is a random set on Ω.

The current investigation focuses on random interval sets rather than point value data. To clarify this we consider an experiment where the exact results, obtained from an accurate and precise digital measuring device, are characterized by a series of distinctive values. The frequency of occurrence for each resulting value is determined by dividing the number of times a particular value occurs by the total number of resulting values in the observation set. Each resulting value is now associated with an inherent frequency of occurrence. [7]

In the previous example the measuring device produced distinctive measurement values; such a device is a rarity in real world experiments and the measuring device often reveals values that are not so specific or perhaps they are between two clearly marked endpoints. A subjective value can be discerned from these endpoints with an associated amount of uncertainty; however, in place of this subjective value the interval A_j can be reported instead.

The resulting interval-valued random set is referred to as a *random interval A*, the universe of discourse is now IR. The random interval *A*, consists of the intervals A_j and their associated frequency of occurrence or weight w_j ($A = \{\langle A_j, w_j \rangle\}$). Each A_j, $1 \leq j \leq n$ is a focal element of the focal set $F = \{A_j\}$). In this investigation the random interval is a left closed, bounded real interval $[l_j, u_j)$, where l_j and u_j are the lower and upper bound values of the random interval. In each interval set, there is an infinite number of possible values (i.e. IR).

The *cardinality* is defined here as the number of possible elements x, within a given subset A and is denoted as |A|. For the purposes of this investigation we limit the interval to discrete intervals with values at every whole number. In other words, the granularity of each interval is 1. Based on this granularity, the number of elements included in interval A_j can be determined by subtracting the lower bound from the upper bound. For instance, a random interval from 8 to 15 ([8, 15)) has a cardinality value of 7, based on the above granularity. Finite discrete intervals are required for the consonant transformation as well as many of the metrics used in this investigation; therefore, only discrete intervals are used.

2.3 Relationships Among Random Intervals

An important characteristic of random sets is *consistency* among its random intervals, it is defined as a common agreement between random intervals; this numerical agreement is referred to as a *core* C. The core of a random interval set is defined mathematically as:

$$\mathrm{C}(A) = \bigcap_{\mathrm{A_j} \in A} \mathrm{A_j} \qquad (2.3.1)$$

where $\in$, signifies "member of." Consider the following intervals: [1, 5), [2, 7) and [3, 5). The intervals overlap from 3 to 5, which forms the core. For a random set to be consistent, a core must exist and thus the expression C must not be equal to the null set.

$$C(A) \neq \emptyset$$

Simply stated the *support* U, is the interval formed by the union of intervals of the random interval set. The support of the random set is defined mathematically as:

$$U(A) = \bigcup_{A_j \in A} A_j \tag{2.3.2}$$

A random interval set is considered *consonant* if nesting exists among the random interval set, (the intervals A_j are nested, set A_1 is included by set A_2, …A_{n-1} is included by set A_n) [2].

$$A_1 \subseteq A_2 \ldots \subseteq A_n$$

Consistency is a weaker property than consonance: each consonant random set is consistent but each consistent set is not necessarily consonant [9].

2.4 Fuzzy Measure

In a *crisp set* (**A**), the elements x are either a members of set **A** or not. Thus, the elements that are members of **A** are assigned a membership value of 1 and non-members are assigned a value of zero. A *fuzzy set* $(\tilde{A})$ is thus a set containing elements that have varying degrees of membership values from 0 to 1 [2].

A *fuzzy measure* is a function $\nu(A) \in [0,1]$ for any interval A. The uncertainty of a fuzzy measure is in the assignment of the membership [2].

2.4.1 Basic Probability Assignment

A function known as a *basic probability assignment* (bpa), [2] "is the degree of belief that a specific element, x, of the universe X belongs to the set A, but not to any specific subset of A." A *power set* is the set of all possible sets of the universe X. In this investigation the bpa is determined based on the frequency of occurrence of the observation interval and is referred to as the weight or evidence w.

$$w_{A_i} = \frac{n_{A_i}}{M} \tag{2.4.1}$$

where, n_{A_i} is the frequency count of set (random interval set) A_i and M is the total number of observation sets (random intervals), $\sum_i w_{A_i} = 1$, $\sum_i n_{A_i} = M$. [2] The weight of the set that contains no random intervals, the null set, is zero and the weight of the set that contains all possible random intervals, is one.

2.4.2 Belief and Plausibility Measures

Consider any interval A, along with its weight w(A). Based on the evidence w, a fuzzy measure known as a belief measure bel(A) is defined as:

$$\mathrm{bel}(A) = \sum_{A_j \subseteq A} w(A_j)\,, \tag{2.4.2}$$

where interval A_j is equivalent to or included in A, $(A_j \subseteq A)$. If $A_j \subseteq A$ is true, then all the evidence supporting A_j is added to the evidence of set A and the sum of these is equal to the belief measure.

The plausibility of A (pl(A)) is [2] the total evidence of A plus the evidence in all the intervals of the universe that intersects $(\cap)$ with A.

$$\mathrm{pl}(A) = \sum_{A_j \cap A \neq \emptyset} w(A_j) \tag{2.4.3}$$

There exists a mutually dual relationship between belief and plausibility, which is described by the following:

$$\mathrm{pl}(B) = 1 - \mathrm{bel}(\overline{B}) \tag{2.4.4}$$

alternatively,

$$\mathrm{bel}(B) = 1 - \mathrm{pl}(\overline{B}) \tag{2.4.5}$$

Where B represents a given interval and $\overline{B}$ represents the complement of that interval. From the above equations it can be shown that whatever the evidence that supports B, the plausibility is at least as great as the belief measure [2], i.e.

$$\mathrm{pl}(B) \leq \mathrm{bel}(B)$$

When the evidence is available only on individual elements x of the universe instead of interval sets A_i, the belief of x equals the plausibility of x. The probability measure is also a frequency of occurrence, but restricted to the individual elements x. The individual elements are all disjoint, and have no intersections; based on this, as well as the definition stated above for the plausibility and belief, the belief equals the plausibility, which equals probability.

When the evidence available is consonant (i.e. nested), then for two sets A and B, which are members of the power set of the universe P(X), we have the following relationships [2].

$$\mathrm{bel}(A \cap B) = \min(\mathrm{bel}(A), \mathrm{bel}(B)) \tag{2.4.6}$$

$$\mathrm{pl}(A \cup B) = \max(\mathrm{pl}(A), \mathrm{pl}(B)) \tag{2.4.7}$$

In the first relationship, the belief measure of the intersection of A and B is the smaller of the belief measure of either A or B. In the second relationship, the plausibility measure of the union of A and B is the larger of the plausibility measure of A or B.

For a consonant body of evidence, the belief and plausibility measures are referred to as necessity N and possibility Π measures, respectively. Again, the mutually dual relationship holds as it did for the plausibility and belief measures; therefore, a necessity measure can be described in terms of a possibility measure. [2]

The plausibility trace ρ_A is simply a "one point coverage" of the plausibility. Defined mathematically as $\rho_A := \mathrm{pl}(\{x\}) = \sum_{x \in A} w_j$, where *A* is a random interval and ρ is a real

number from 0 to 1 [8]. In other words, the plausibility trace is a mapping of the plausibility of the singleton elements, x of the random set *A* to the unit interval. For consonant sets the plausibility trace becomes the possibility distribution.

2.4.3 Possibility Measure

Lets begin with a possibility distribution r_x (or π), which is a mapping of the singleton elements, x of random interval set *A* to the unit interval. From the possibility distribution π, is a weak likelihood of events can be described by the use of a set function called the possibility measure Π [7]. A fuzzy measure is a possibility measure when [4]: (1) the possibility measure of the null set is zero, (2) for all A and B members of universe X, and if A is included or equal to set B then the possibility measure of A is less than or equal to that of B, and (3) for a nested sequence of sets A_i to A_n the possibility measure of A_i is the largest of the possibility measure of the sets included in A_i.

(1) $\Pi(\emptyset) = 0$

(2) $\forall A, B \in X,$ if $A \subseteq B$ $\Pi(A) \leq \Pi(B)$ and,

(3) For any increasing sequence $A_1 \subseteq A_2 \ldots \subseteq A_n$, $\Pi(A_i) = \sup_{i \in I} \Pi(A_i)$

Consider A_i, a member of *A*, the possibility measure is the largest of the possibility distribution in A_i (given that *A* is consonant),

$$\Pi(A_i) = \sup(r(x) \mid x \in A_i) . \tag{2.4.8}$$

When $\Pi(A_i) = 1$, it is totally possible the true value of x is in A_i. Similarly when $\Pi(A_i) = 0$ it is impossible that the true value of x is in A_i.

2.5 Normalization Algorithms

The focus of this investigation deals with two algorithms: (1) proposed by Ross and Donald [4] that forms a consonant set from non-consonant interval data, and (2) proposed by Joslyn [9] that forms a consistent set from an inconsistent random set.

2.5.1 Consonant Transformation

For a consonant interval set, a normal possibility distribution is readily obtained of which possibility measures are thus determined. A non-consonant set does not naturally facilitate the determination of a normal possibility distribution. A consonant interval set is thus desired and the algorithm proposed by Ross and Donald [4] transforms a non-consonant random set of data to a consonant set of data, a normal possibility distribution is then obtained for the consonant random set. Their algorithm is summarized below.

Suppose that a system is described within a domain $\Omega = \{w_1, w_2, \ldots w_n\}$, the behavior is described by evidence obtained over a collection of sets, A. Where A is a subset of Ω. This collection of evidence forms a vector $\vec{F}$, comprised of the observation set and its

weight. Suppose, it is determined that $\vec{F}$ is non-consonant and that we wish to transform it to a consonant set. The first step, is to determine the intersections of the various intervals $A_i \cap A_j = B_{ij}$. The smallest of these intervals has the most intersections with the observations and the next smallest is the one with the second most intersections, and so forth. Also included in this set is an interval equivalent to the support, if it is not already present. This interval is composed of the minimum lower-bound value and the maximum upper-bound value of the observation set. Hence, the algorithm is also applicable to disjoint sets as well as other non-consonant interval sets (i.e. consistent but non-consonant, overlapping but non-consistent).

The weight of these focal elements is based on the minimum norm, $v_{ij} = \min\left(w_{A_i}, w_{A_j}\right)$. A minimum norm is based on non-additivity and the weights do not necessarily add to one; therefore, the weights derived are normalized as,

$$\eta = \frac{v_{ij}}{\sum_{i=1}^{Q} v_{ij}} \tag{2.5.1}$$

The weight of the interval comprising the support, if not already present, is based on the minimum of the weight present in the observation set. This process yields a vector $\vec{G} = \left\langle (B_{ij}, v_{ij}) \right\rangle$ of comprised of the intersections and their respective weights, some of which are consonant with their parent set while not necessarily consonant with other parent sets. The total number of focal elements in $\vec{G}$ is denoted as Q, the number elements in the designated consonant set H is denoted as Q_H, and the number of elements in the designated non-consonant set I is denoted as Q_I. The focal elements of G compose

the focal set. In this step the analyst's subjectivity is required if there are more than one group of possible choices to place in the consonant set.

$\vec{G}$ also contains a min-max interval which is composed of the minimum lower bound and the maximum upper bound of *A*. This additional interval is included if no equivalent interval is present as a random interval. Due to an increase in intervals, the sum of the weight for the intervals in $\vec{G}$ may no longer be normal; therefore, the weights must be normalized. (Keep in mind that a simpler $\vec{G}$ can be also be formed without determining the intersections and only considering the focal elements.) The elements (intervals) are then partitioned into a consonant set H and a non-consonant set I where the focal elements of H and I are denoted H_i and I_j, respectively. The normalized weight of the focal elements in set I are distributed to the random interval observations in H.

The redistribution of weight is based on the dissonance between H_i and I_j, where dissonance is the conflict between the two sets. The lower the dissonance the higher the similarity β. Similarity values are calculated for each H_i of H. This is accomplished by determining the number of common elements between the H_i of H and the I_j of I and dividing it by the cardinality of the each H_i, denoted $|H_i|$.

$$\beta_{ij} = \frac{|H_i \cap I_j|}{|H_i|} \tag{2.5.2}$$

Each H_i has Q similarity values, in other words there are $Q_H * Q_I$ total similarity values. For example, if there are 4 intervals in H and 2 in I then the total number of similarity

values is 2 * 4 = 8 or two similarity values for each H_i in Q_H. With β the redistribution factors k, are calculated. This is accomplished with the following formula:

$$k_{ij} = \frac{\beta_{ij}}{\sum_{i=1}^{Q_H} \beta_{ij}} \tag{2.5.3}$$

k_{ij} when multiplied by the normalized weight of I_j produces a fraction of weight that is distributed to the weight of set H_i. The redistributed weight of H_i (m) is calculated by adding the normalized weight of H_i in the consonant set H to the weight being distributed from I_j of the non-consonant set I. For each I_j, there is a fraction of weight that is distributed to each H_i.

$$m(H_i) = \eta(H_i) + \sum_{j=1}^{Q_i} k_{ij} * \eta(I_j) \tag{2.5.4}$$

From the resulting consonant set and their final weights, a normal possibility distribution and measures can be determined.

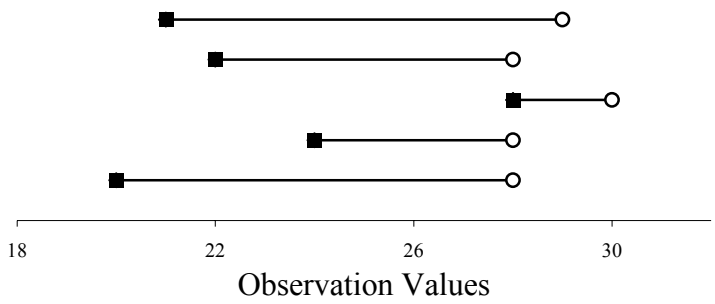
2.5.2 Consonant Transformation Illustrative Example

Donald and Ross [4] present the following example for consonant transformation.
The observation record is found in the 1st column and the determined weights are found in the 2nd column:

Table 2.5a

Interval	Weight
[20, 28)	0.2
[24, 28)	0.2
[28, 30)	0.2
[22, 28)	0.2
[21, 29)	0.2

Figure 2.5a



The above intervals are arbitrarily subdivided into two sets, one composed of a consonant set and the other a non-consonant set. Included in the consonant set is an interval that includes all the original intervals. In this particular example there are 2 options for normalizing the non-consonant set, however we continue with Option 1 (see Chapter 3 for Option 2).

Table 2.5b

Option 1		Option 2	
H_i	I_j	H_i	I_j
[20, 30)	[20, 28)	[20, 30)	[20, 28)
[21, 29)	[28, 30)	[20, 28)	[21, 29)
[22, 28)		[22, 28)	
[24, 28)		[24, 28)	

Since there are only five unique intervals in the original data, then the total weight for each arbitrary interval is 1/5 or 0.2. Note, that an interval including the support is included and its weight is the minimum weight of the intervals included in the original set. Each interval occurs only once in the collection of six arbitrary divided intervals (G-set) thus, each has a normalized weight of 1/6.

Table 2.5c

G – Sets	Interval	Total Weight	Normalized Weight
$Q_H = 4$	[20, 30)	0.2	1/6
	[21, 29)	0.2	1/6
	[22, 28)	0.2	1/6
	[24, 28)	0.2	1/6
$Q_I = 2$	[20, 28)	0.2	1/6
	[28, 30)	0.2	1/6

There are 4 consonant intervals (H_i), i =1, 2, 3, 4 and 2 non-consonant intervals (I_j), j =1, 2. See table 3.2e.

Table 2.5d

H_i	I_j
[20, 30)	[20, 28)
[21, 29)	[28, 30)
[22, 28)	
[24, 28)	

β values are now calculated.

(1) $$\beta_{11} = \frac{|[20,30) \cap [20,28)|}{|[20,30)|} = \frac{8}{10} = 0.800$$

(2) $$\beta_{12} = \frac{|[20,30) \cap [28,30)|}{|[20,30)|} = \frac{2}{10} = 0.200$$

(3) $$\beta_{21} = \frac{|[21,29) \cap [20,28)|}{|[21,29)|} = \frac{7}{8} = 0.875$$

(4) $$\beta_{22} = \frac{|[21,29) \cap [28,30)|}{|[21,29)|} = \frac{1}{8} = 0.125$$

(5) $$\beta_{31} = \frac{|[22,28) \cap [20,28)|}{|[22,28)|} = \frac{6}{6} = 1.000$$

(6) $$\beta_{32} = \frac{|[22,28) \cap [28,30)|}{|[22,28)|} = \frac{0}{6} = 0$$

(7) $$\beta_{41} = \frac{|[24,28) \cap [20,28)|}{|[24,28)|} = \frac{4}{4} = 1.000$$

(8) $$\beta_{42} = \frac{|[24,28) \cap [28,30)|}{|[24,28)|} = \frac{0}{4} = 0$$

From the β values the redistribution factors (k) are calculated.

(9) $$k_{11} = \frac{\beta_{11}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.218$$

(10) $$k_{12} = \frac{\beta_{12}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 0.615$$

(11) $$k_{21} = \frac{\beta_{21}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.288$$

(12) $$k_{22} = \frac{\beta_{22}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 0.385$$

(13) $$k_{31} = \frac{\beta_{31}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.272$$

(14) $$k_{32} = \frac{\beta_{32}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 0$$

(15) $$k_{41} = \frac{\beta_{41}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.272$$

(16) $$k_{42} = \frac{\beta_{42}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 0$$

The final weight for each H_i is now calculated using the formula below.

(17) $$m(H_i) = \eta(H_i) + \sum_{j=1}^{Q_i} k_{ij} * \eta(I_j)$$

A_i H_i

A_1 [20, 30) $$m = 0.167 + (0.218 + 0.615) * 0.167 = \frac{1.833}{6} = 0.306$$

A_2 [21, 29) $$m = 0.167 + (0.238 + 0.385) * 0.167 = \frac{1.623}{6} = 0.270$$

A_3 [22, 28) $$m = 0.167 + (0.272 + 0) * 0.167 = \frac{1.272}{6} = 0.212$$

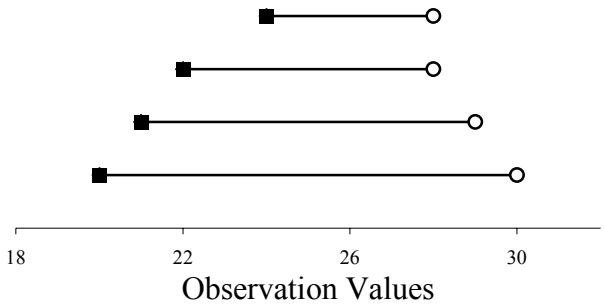
A_4 [24, 28) $$m = 0.167 + (0.272 + 0) * 0.167 = \frac{1.272}{6} = 0.212$$

$$\sum = 1.00$$

Table 2.5e

Random Interval	Weight
[20, 30)	0.306
[21, 29)	0.270
[22, 28)	0.212
[24, 28)	0.212

Figure 2.5b



2.5.3 Consistent Transformation

From a consistent random set a possibility distribution can be obtained and consequently possibility measures [8]. Possibility measures can be derived from a consistent but non-consonant set, however it is more natural to use a consonant set [4]. Joslyn proposed an algorithm for transforming an inconsistent random set to a consistent set. A summary of this approach, as employed in this investigation, is given below. Only the intervals formed from the support of the original set are considered as normalizing intervals. First the endpoints of the intervals of the random set are placed into an array $\dot{E}$, the array is then ordered from smallest to largest, and finally the duplicate endpoints are removed from the array forming E of non duplicate endpoints (e_1, e_2, e_3, $e_4 \ldots e_n$). From E a class $\Gamma = \{G_k\}$, is formed which is the finest partition of the support of the random set. G_k are the intervals formed from the endpoints in array E ($G_1 = [e_1, e_2)$, $G_2 = [e_2, e_3) \ldots Gn = [e_{n-1}, e_n)$). The plausibility of each interval of Γ is found using the plausibility assignment

(plausibility trace), which is a one point coverage or the trace of the plausibility over S (support of *A*). For further details on this method see [8, 9].

For an inconsistent random set, the plausibility trace is subnormal (the max value is less than one); for a consistent or consonant set the plausibility trace is normal (max value is equal to one). In a consonant or consistent set the plausibility trace will achieve a value equal to one for the point or interval encompassing the core.

The inconsistent set is made consistent by selecting a normalizing interval from Γ to transform the inconsistent set. The operations on the G_k portions are different from the operations on focal sets of the random set due to the linear ordering of the real line IR. Therefore, the G_k's are utilized in the following operations. A concatenation operation is used to unite two intervals to form set I, which result in an ordered sequence of G_k's (see [7, 8]).

$$I = \dot{\bigcup}_{k=I_l}^{I_u} G_k \tag{2.5.5}$$

The concatenation $\dot{\bigcup}$ of the normalizing interval G_K and the focal sets of *A* to form the transformed consistent random intervals $\hat{A}_j$.

$$\hat{A}_j := A_j \dot{\bigcup} G_K \tag{2.5.6}$$

As stated above, only the elements that make up Γ are considered as possible normalizing intervals. The weight of these consistent intervals $\hat{m}^K(I)$, is determined using the following proposition.

$$\hat{m}^K(I) = \sum_{k=I_l}^{I_u} m(G_{I_l} G_{I_l+1} \cdots G_k), \quad I_u = K, \text{ Case 1}$$

$$\hat{m}^K(I) = \sum_{k=I_l}^{I_u} m(G_k G_{k+1} \cdots G_{I_u}), \quad I_l = K, \text{ Case 2}$$

$$\hat{m}^K(I) = m(I), \qquad I_l < K < I_u, \text{ Case 3}$$
$$\hat{m}^K(I) = 0, \qquad \text{otherwise, Case 4}$$

From the evidence the transformed plausibility assignment is determined for each of the normalized intervals that make up the normalized focal set. The possibility histogram and consequently a possibility distribution can be determined from the consistent normalized set.

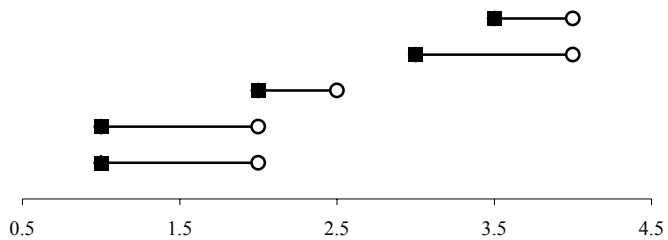
2.5.4 Consistent Transformation Illustrative Example

The following example is illustrated by Joslyn [9], given the following observation record,

Table 2.5f

Interval
[1, 2)
[1, 2)
[2, 2.5)
[3, 4)
[3.5, 4)

Figure 2.5c



Observation Values

It is determined to be non-consistent and a consistent interval set is desired. The resulting random interval set is as follows.

Table 2.5g

Random Interval	Weight
[1, 2)	0.4
[2, 2.5)	0.2
[3, 4)	0.2
[3.5, 4)	0.2

As stated above, only the intervals G_k from the class $\Gamma = \{G_k\}$, are considered as normalizing intervals. The following are the available normalizing intervals:

Table 2.5h

G_k	Partition	Plausibility Trace
G_1	[1, 2)	0.4
G_2	[2, 2.5)	0.2
G_3	[2.5, 3)	0.2
G_4	[3, 3.5)	0.2
G_5	[3.5, 4)	0.4

Consider G_4 as the normalizing interval. Table 2.5i contains the normalized intervals obtained from the concatenation step.

$$\hat{A}_j := A_j \dot{\cup} G_K$$

Table 2.5i

Interval ($\hat{A}_j$)
[1, 3.5)
[2, 3.5)
[3, 4)
[3, 4)

The weight is obtained using the following proposition:

$\hat{m}^K(I) = \sum_{k=I_l}^{I_u} m(G_{I_l} G_{I_l+1} \cdots G_k)$, $I_u = K$, Case 1

$\hat{m}^K(I) = \sum_{k=I_l}^{I_u} m(G_k G_{k+1} \cdots G_{I_u})$, $I_l = K$, Case 2

$\hat{m}^K(I) = m(I)$, $I_l < K < I_u$, Case 3

$\hat{m}^K(I) = 0$, otherwise, Case 4

Keep in mind that the intervals $\hat{A}_j$, are an ordered sequence of G_k's, $I = \dot{\cup}^{Iu}_{k=I_l} G_k$.

For I = [1, 3.5), K = 4 = I_u, case 1 is applied as follows

$\hat{m}^K(I) = m(G_1) + m(G_1G_2) + m(G_1G_2G_3) + m(G_1G_2G_3G_4)$

$= m([1, 2)) + m([1, 2.5)) + m([1, 3)) + m([1, 3.5))$

$= .4 + 0 + 0 + 0 = 0.4$

For I = [3, 4), K = 4 = I_l, case 2 is applied as follows

$\hat{m}^K(I) = m(G_5) + m(G_4G_5)$

$= m([3.5, 4)) + m([3, 4))$

$= 0.2 + 0.2 = .4$

For I = [2, 3.5), K = 4 =I_u, case 1 is applied as follows

$\hat{m}^K(I) = m(G_2) + m(G_2G_3) + m(G_2G_3G_4)$

= m([2, 2.5)) + m([2, 3)) + m([2, 3.5))

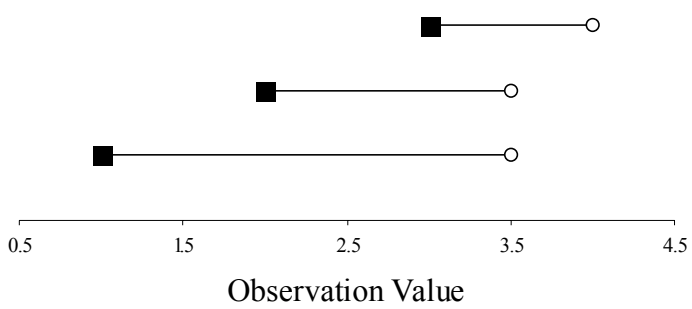
= 0.2 + 0 + 0 = 0.2

Thus, the transformed consistent set is as follows.

Table 2.5j

Random Interval	Weight
[1, 3.5)	0.4
[2, 3.5)	0.2
[3, 4)	0.4

Figure 2.5d



2.6 The "Best" Alternative

As human beings, we constantly encounter situations where we must make a decision from a given set of alternatives. Each alternative may pose a different outcome; it is often beneficial to base our decision on the intended outcome of the alternative. The same can be said for the situation encountered in this investigation. In the Ross & Donald algorithm [4], different possibility histograms and thus possibility distributions

result when different groups of intervals are placed in the consonant set H and non-consonant set I. The same is true for Joslyn's algorithm [8]; different normalizing intervals may produce different possibility histograms. Which of the resulting transformed sets is the "best" alternative? An obvious conjecture would be the transformed set that preserves and highly reflects the information of the original set.

The resemblance between the transformed set and the original set can be determined by comparing the plausibility trace of the original set and the plausibility trace of the transformed set. This can be done using any of the various distance or similarity metrics available. The plausibility trace of a consonant set is equivalent to the possibility histogram and thus possibility distribution. For a consistent but not consonant set, the plausibility distribution is equivalent to the possibility histogram from which the possibility distribution can be obtained (see [9]). It may also be beneficial to compare the overall uncertainty of the transformed set. According to Klir and Wierman [6], only three types of uncertainties are recognized in the available theories of classical set theory, fuzzy set theory, probability theory, possibility theory, and evidence theory. These uncertainty types are: fuzziness (or vagueness), which is associated in imprecision of the boundaries; nonspecificity (or imprecision), which is associated with the size of the available alternatives in the set; and strife (or discord), which is associated with conflict among data [6].

2.6.1 Resemblance

In this investigation the resemblance or closeness between the plausibility trace of the original set and the plausibility trace of the transformed set is measured by using the Hamming distance [5] and a similarity metric (which uses the inner and outer product)[2]. Recall, that the granularity is equal to 1, this produces discrete sets and allows the application of Hamming Distance metric and Similarity metric.

The Hamming distance metric, measures the distance between two membership functions, or in this investigation between two plausibility traces $\rho_{\tilde{A}}$ and $\rho_{\tilde{B}}$.

$$f(A) = \sum_{x \in X} \left| \rho_{\tilde{A}}(x) - \rho_{\tilde{B}}(x) \right| . \tag{2.6.1}$$

Where the smaller the value of distance produced the closer the plausibility traces are. Recall that for consistent sets, the plausibility trace becomes a possibility distribution.

For the similarity metric, either the average or the minimum value of the inner product of the fuzzy vectors $\vec{\tilde{A}}$ and $\vec{\tilde{B}}$, and the outer product of the complement of $\vec{\tilde{A}}$ and $\vec{\tilde{B}}$, is determined (here the former is used). In this investigation, the fuzzy vector $\vec{\tilde{A}}$ is composed of the plausibility trace values for members of the fuzzy set $\tilde{A}$, whereas the complement of the fuzzy vector $\vec{\tilde{A}}$ denoted $\overline{\vec{\tilde{A}}}$ is composed of the membership values $\rho_{\overline{\tilde{A}}}$ for the members of the complement. Since the fuzzy vectors, plausibility trace vector

$(\vec{\tilde{A}})$ and possibility distribution $(\vec{\tilde{B}})$, being compared are taken from a discrete set we will denote the vectors simply as $\tilde{A}$ and $\tilde{B}$. The inner product of $\tilde{A}$ and $\tilde{B}$ is defined as follows:

$$\tilde{A} \bullet \tilde{B}^T = \max_{i=1}^{n}(\min(\rho_{\tilde{A}}(x_i), \rho_{\tilde{B}}(x_i))). \tag{2.6.2}$$

The outer product of the compliment is defined as follows:

$$\overline{\tilde{A}} \oplus \overline{\tilde{B}}^T = \min_{i=1}^{n}(\max(\rho_{\overline{\tilde{A}}}(x_i), \rho_{\overline{\tilde{B}}}(x_i))). \tag{2.6.3}$$

The similarity measure is defined as

$$(\tilde{A}, \tilde{B})_2 = \frac{1}{2}\left[(\tilde{A} \bullet \tilde{B}) + (\overline{\tilde{A}} \oplus \overline{\tilde{B}}^T)\right] \tag{2.6.4}$$

When this value approaches 1, the two fuzzy sets are "more closely similar;" and when the value approaches 0, the two are "more far apart" (dissimilar) [2].

2.6.2 Measures of Uncertainty

The amount of uncertainty in evidence is useful in determining which set of information has the least conflict among its members, and which set is the most specific. In evidence, the lower the value of Nonspecificity the more specific the information is. In the manner, the lower the value of Strife the lower the conflict among members of the set.

Klir and Weirman [6] describe one form of uncertainty emerging in evidence

"whenever we know some alternative of interest belongs to a particular set of alternatives, but we do not know which one in the set it is (we do not know which alternative it is). To identify this alternative in question, we need information by which the uncertainty is completely removed. The amount of uncertainty associated with a set of alternatives may thus be measured by the amount of information needed to remove the uncertainty."

A measure of this type of uncertainty in finite crisp sets (classical theory), was first proposed by Hartley [6]. This measure is determined through a function known as the Hartley function (measured in bits).

$$H(S_R) = \log_2 |S_R| \tag{2.6.5}$$

The Hartley function, is applicable to a random set S_R where the variable is represented as individual elements x rather than interval sets (probabilities).

U-uncertainty, proposed by Higashi and Klir [6], is used to measure the same type of uncertainty as Hartley but for fuzzy sets or possibility distributions. For any normal fuzzy set ($\widetilde{A}$) defined on a finite universe by using the α-cuts of F and the cardinality of the α-cuts.

$$U(\widetilde{A}) = \int_0^1 \log_2 |{}^{\alpha}\widetilde{A}| \, d\alpha \tag{2.6.6}$$

A possibility distribution is a membership function of some fuzzy set [7]; therefore, this form of U-uncertainty may also be applicable to possibility distributions (possibility histograms). This measurement of uncertainty can also be applied to evidence theory and

is denoted N-uncertainty [6]. This measure uses available evidence of the random intervals A and its cardinality

$$N(m) = \sum_{A \in S_R} w(A) \log_2 |A| \tag{2.6.7}$$

where <A, w> is an arbitrary body of evidence. Klir and Weirman [6] view this function as a reasonable measure of nonspecificity.

Conflict among a body of evidence arises when the focal elements of the body of evidence do not necessarily agree with one another. Shannon proposed a function (denoted Shannon entropy) that measures this uncertainty; however, it is only applicable to probability measures [6]. In evidence theory, there is no generalized counterpart to Shannon entropy. Some function for the measurement of this uncertainty in evidence have been proposed, such as: conflict C(w), dissonance E(w), discord D(w), and strife St(w) [6]. The function E is defined by the following formula

$$E(w) = -\sum_{A \in F} w(A) \log_2 pl(A). \tag{2.6.8}$$

For any consistent focal set, the value of E will be zero, the plausibility of each focal set will be one and thus $\log_2 1 = 0$. C is a better measure of conflicting evidence, however, it does not properly scale each particular conflict between two focal sets A and B [6].

$$C(w) = -\sum_{A \in F} w(A) \log_2 Bel(A) \tag{2.6.9}$$

This drawback is overcome with the function D,

$$D(w) = -\sum_{A \in F} w(A) \log_2 \sum_{B \in F} w(B) \frac{|A \cap B|}{|B|}. \tag{2.6.10}$$

However, a defect was apparent with this function when dealing with focal set A being contained within focal set B ($A \subset B$); w(B) is taken to be in conflict with w(A) to a degree |B-A|/|B|. B implies A and there should be no conflict present between A and B since A is contained in B. Thus, strife has been proposed as a better measure of conflict [6],

$$ST(w) = -\sum_{A \in F} w(A) \log_2 \sum_{B \in F} w(B) \frac{|A \cap B|}{|A|} \tag{2.6.11}$$

2.6.3 Aggregate Uncertainty

It would be beneficial to obtain a measure of the combined uncertainty (nonspecificity and conflict). Unfortunately merely adding these results of uncertainty is questionable (see [10]); as a result, a function has been proposed to measure aggregate uncertainty (AU) in evidence. This function is supposed to capture both nonspecificity and conflict. An algorithm has been developed from the belief function to calculate aggregate uncertainty and the steps taken are repeated here [6].

Algorithm 2.6.1

Input: For a body of evidence defined on X, and the belief measure bel on X.

Output: $AU(bel) = -\sum_{x \in X} p_x \log_2 p_x$, $p_i \geq 0$, $\sum_{x \in X} p_x = 1$, and $bel(A) \leq \sum_{x \in A} p_x$ for all $\varnothing \neq A \subseteq X$.

Step 1. Find a non-empty set $A \subseteq X$, such that $\frac{bel(A)}{|A|}$ is maximal. If there are more than one such sets, take the one with the maximal cardinality |A|.

Step 2. For $x \in A$, put $p_x = \dfrac{bel(A)}{|A|}$.

Step 3. For each $B \subseteq X - A$, put $bel(B) = bel(B \cup A) - bel(B)$.

Step 4. Put $X = X - A$.

Step 5. If $X \neq \varnothing$ and $bel(X) > 0$, then go to step 1.

Step 6. If $bel(X) = 0$ and $X \neq \varnothing$, then put $p_x = 0$ for all $x \in X$.

Step 7. Calculate $AU(bel) = -\sum_{x \in X} p_x \log_2 p_x$

When dealing with a consonant body of evidence the algorithm above can be substantially simplified using the possibility distribution [6]. The simplified algorithm is repeated here ($\pi = r$, possibility distribution).

Algorithm 2.6.2

Input: $\boldsymbol{\pi} = \langle \pi_1, \pi_2, \ldots \pi_n \rangle$, where n is a member of the set of positive integers.

Output: $AU(\Pi)$, for p_i where i is a member of the set of integers from 1 to n, such that

$AU(\Pi) = -\sum p_i \log_2 p_i$, with $p_i \geq 0$ for all i from 1 to n, $\sum_{i=1}^{n} p_i = 1$, and

$\sum_{x \in A} p_x \leq \Pi(A)$ for all $\varnothing \neq A \subset X$. Where is the possibility measure of focal set A.

Step 1. Let j = 1 and $\pi_{n+1} = 0$.

Step 2. Find maximal $i \in \{j, j+1, \ldots, n\}$ such that $\dfrac{\pi_j - \pi_{i+1}}{i+1-j}$ is maximal.

Step 3. For $k \in \{j, j+1, \ldots, i\}$ put $p_k = \dfrac{\pi_j - \pi_{i+1}}{i+1-j}$.

Step 4. Put j = i + 1.

Step 5. If $i < n$, then go to step 2.

Step 6. Calculate $AU(\Pi) = -\sum_{i=1}^{n} p_i \log_2 p_i$

In an effort to compare the resulting transformed sets, metrics of inherent uncertainty must be used.

2.6 Chapter Summary

This chapter introduces and describes the terms, properties, algorithms, similarity, distance, and uncertainty metrics used or developed in this investigation. The chapter begins by defining a random set and other terms associated with a random set such as random interval, weight, and cardinality. Properties such as consistency and consonance are also defined, as well as the core and the support of a random set. A fuzzy measure is introduced and it is shown that plausibility, belief, probability and possibility measures are special types of fuzzy measures. The chapter goes on to discuss the two transformation algorithms used in this investigation; the first transforms a non-consonant interval set to a consonant set and the second transforms an inconsistent interval set to a consistent interval set. Uncertainty measures are used to assess the resulting transformed sets. This chapter serves to acquaint the reader with essential information related to this study.

Chapter 3

Consonant and Consistent Transformation Algorithms

3.1 Chapter Introduction

The transformation algorithms proposed by Ross & Donald [4] and Joslyn [7,8] serve as tools to transform a non-consonant random set to a consonant random set or inconsistent random set to a consistent set, respectively. Software has been developed using the C++ language that incorporates these algorithms and allows the transformed sets to be obtained more readily and thus make this study more efficient. The following chapter demonstrates the utility of these algorithms along with some illustrative examples.

3.2 Possibility Histograms

Joslyn [7] describes a possibilistic histogram as a possibility distribution derived from consistent empirical random intervals. They are derived from overlapping interval observations and governed by the mathematics of random sets. All possibilistic

histograms are fuzzy intervals. The desired outcome of the above transformation algorithms is a normal possibility distribution.

3.3 Transformation Algorithms

As mentioned previously, the transformation algorithms in this study serve to transform non-consonant or inconsistent random sets to more useful form. A possibility histogram and thus possibility measures can be obtained from the consistent random set [9]. The same is true for a consonant set (also consistent); however, the possibility measure is obtained more naturally [4].

3.3.1 Software

The software developed herein calculates the plausibility trace values over the support for a given interval set of data. The original data record is all that the software requires. The original interval set may be inconsistent, inconsistent and disjoint, non-consonant, or consonant. For an inconsistent and disjoint interval set, the software defaults to the consistent transformation algorithm; for an inconsistent but overlapping interval set, the software prompts the user to select which transformation approach to utilize; for a consistent set, the software defaults to the consonant transformation algorithm; finally for a consonant interval set, the software readily obtains the plausibility trace distribution from which the possibility distribution is obtained.

3.3.2 Consistent Transformation

As mentioned in Chapter 2, consistent transformation is made possible through the use of a designated normalizing interval G_K. Based on G_K, the set of normalized intervals and their weights are obtained. The consistent transformation algorithm is demonstrated here with the following example proposed by Joslyn [8]. The intervals are "left end closed" and continuous.

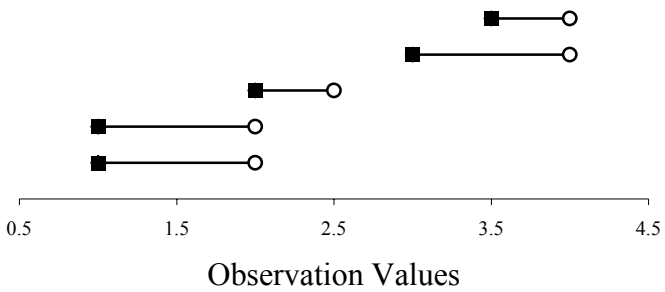
Example 3.1

The following example is illustrated in Chapter 2 however in this case a different normalizing interval is used. Given the following measurement record consisting of only the observation interval sets, the weight for each interval set is calculated based on the frequency of occurrence of each interval (see Chapter 2).

Table 3.1a

Interval
[1, 2)
[1, 2)
[2, 2.5)
[3, 4)
[3.5, 4)

Figure 3.1a



The resulting random interval set is as follows.

Table 3.1b

Random Interval	Weight
[1, 2)	0.4
[2, 2.5)	0.2
[3, 4)	0.2
[3.5, 4)	0.2

The above set contains no core and thus is inconsistent; however, it can be transformed to a consistent set using a normalizing interval. As stated in Chapter 2, only the intervals G_k from the class $\Gamma = \{G_k\}$, were considered as normalizing intervals. The following are considered as normalizing intervals:

Table 3.1c

G_k	Partition	Plausibility Trace
G_1	[1, 2)	0.4
G_2	[2, 2.5)	0.2
G_3	[2.5, 3)	0.2
G_4	[3, 3.5)	0.2
G_5	[3.5, 4)	0.4

Consider G_3 as the normalizing interval. Table 3.1d contains the normalized intervals obtained from the concatenation step.

$$\hat{A}_j := A_j \dot{\cup} G_K$$

Table 3.1d

Interval ($\hat{A}_j$)
[1, 3)
[2, 3)
[2.5, 4)
[2.5, 4)

The weight is obtained using the following proposition:

$\hat{m}^K(I) = \sum_{k=I_l}^{I_u} m(G_{I_l} G_{I_l+1} \cdots G_k)$, $I_u = K$, Case 1

$\hat{m}^K(I) = \sum_{k=I_l}^{I_u} m(G_k G_{k+1} \cdots G_{I_u})$, $I_l = K$, Case 2

$\hat{m}^K(I) = m(I)$, $I_l < K < I_u$, Case 3

$\hat{m}^K(I) = 0$, otherwise, Case 4

Keep in mind that the intervals $\hat{A}_j$, are an ordered sequence of G_k's, $I = \dot{\bigcup}^{Iu}_{k=I_l} G_k$.

For I = [1, 3), K = 3 = I_u, case 1 is applied as follows

$\hat{m}^K(I) = m(G_1) + m(G_1G_2) + m(G_1G_2G_3)$

$= m([1, 2)) + m([1, 2.5)) + m([1, 3))$

$= 0.4 + 0 + 0 = 0.4$

For I = [2, 3), K = 3 = I_u, case 1 is applied as follows

$\hat{m}^K(I) = m(G_2) + m(G_2G_3)$

$= m([2, 2.5)) + m([2, 3))$

$= 0.2 + 0 = .2$

For I = [2.5, 4), K = 3 =I_l, case 2 is applied as follows

$\hat{m}^K(I) = m(G_5) + m(G_4G_5) + m(G_3G_4G_5)$

$= m([3.5, 4)) + m([3, 4)) + m([2.5, 4))$

$= 0.2 + 0.2 + 0 = 0.4$

The software follows the above procedure (given a measurement record) to obtain the normalized consistent random set included in the following table:

Table 3.1e

Random Interval	Weight
[1, 3)	0.4
[2, 3)	0.2
[2.5, 4)	0.4

The software goes on to obtain the plausibility trace values for the above set and sends these values to a user designated data file for plotting the plausibility trace (possibility histogram for consistent sets). The core of this random set is [3, 3.5).

3.3.3 Consonant Transformation

The following example was presented in Chapter 2, and is repeated here however Option 2 is used.

Example 3.2

The observation record (required for use in the software) is found in the 1st column and the determined weights are found in the 2nd column:

Table 3.2a

Interval	Weight
[20, 28)	0.2
[24, 28)	0.2
[28, 30)	0.2
[22, 28)	0.2
[21, 29)	0.2

The above intervals are arbitrarily subdivided into two sets, one composed of a consonant set and the other a non-consonant set. Included in the consonant set is an interval that includes all the original intervals. In this particular example there are 2 options for normalizing the non-consonant set, however we continue with Option 2 (see Chapter 2, Section 2.5.2 for Option 1).

Table 3.2b

Option 1		Option 2	
H_i	I_j	H_i	I_j
[20, 30)	[20, 28)	[20, 30)	[20, 28)
[21, 29)	[28, 30)	[20, 28)	[21, 29)
[22, 28)		[22, 28)	
[24, 28)		[24, 28)	

Since there were only five unique intervals in the original data, then the total weight for each arbitrary interval is 1/5 or 0.2. Each interval occurs only once in the collection of six arbitrary divided intervals (G-set) thus, each has a normalized weight of 1/6.

Table 3.2c

G – Sets	Interval	Total Weight	Normalized Weight
$Q_H = 4$	[20, 30)	0.2	1/6
	[20, 28)	0.2	1/6
	[22, 28)	0.2	1/6
	[24, 28)	0.2	1/6
$Q_I = 2$	[21, 29)	0.2	1/6
	[28, 30)	0.2	1/6

There are 4 consonant intervals (H_i), i =1, 2, 3, 4 and 2 non-consonant intervals (I_j), j =1, 2. See table 3.2e.

Table 3.2d

H_i	I_j
[20, 30)	[21, 29)
[20, 28)	[28, 30)
[22, 28)	
[24, 28)	

β values are now calculated.

(1) $$\beta_{11} = \frac{|[20,30) \cap [21,29)|}{|[20,30)|} = \frac{8}{10} = 0.800$$

(2) $$\beta_{12} = \frac{|[20,30) \cap [28,30)|}{|[20,30)|} = \frac{2}{10} = 0.200$$

(3) $$\beta_{21} = \frac{|[20,28) \cap [21,29)|}{|[20,28)|} = \frac{7}{8} = 0.875$$

(4) $$\beta_{22} = \frac{|[20,28) \cap [28,30)|}{|[20,28)|} = \frac{0}{8} = 0$$

(5) $$\beta_{31} = \frac{|[22,28) \cap [21,29)|}{|[22,28)|} = \frac{6}{6} = 1.000$$

(6) $$\beta_{32} = \frac{|[22,28) \cap [28,30)|}{|[22,28)|} = \frac{0}{6} = 0$$

(7) $$\beta_{41} = \frac{|[24,28) \cap [21,29)|}{|[24,28)|} = \frac{4}{4} = 1.000$$

(8) $$\beta_{42} = \frac{|[24,28) \cap [28,30)|}{|[24,28)|} = \frac{0}{4} = 0$$

From the β values the redistribution factors (k) are calculated.

(9) $$k_{11} = \frac{\beta_{11}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.218$$

(10) $$k_{12} = \frac{\beta_{12}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 1$$

(11) $$k_{21} = \frac{\beta_{21}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.238$$

(12) $$k_{22} = \frac{\beta_{22}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 0$$

(13) $$k_{31} = \frac{\beta_{31}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.272$$

(14) $$k_{32} = \frac{\beta_{32}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 0$$

(15) $$k_{41} = \frac{\beta_{41}}{\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41}} = 0.272$$

(16) $$k_{42} = \frac{\beta_{42}}{\beta_{12} + \beta_{22} + \beta_{32} + \beta_{42}} = 0$$

The final weight for each H_i is now calculated using the formula below.

(17) $m(H_i) = \eta(H_i) + \sum_{j=1}^{Q_i} k_{ij} * \eta(I_j)$

A_i	H_i	
A_1	[20, 30)	$m = 0.167 + (0.218 + 1) * 0.167 = \frac{2.218}{6} = 0.370$
A_2	[20, 28)	$m = 0.167 + (0.238) * 0.167 = \frac{1.238}{6} = 0.206$
A_3	[22, 28)	$m = 0.167 + (0.272 + 0) * 0.167 = \frac{1.272}{6} = 0.212$
A_4	[24, 28)	$m = 0.167 + (0.272 + 0) * 0.167 = \frac{1.272}{6} = 0.212$
		$\sum = 1.00$

Using the following equation the plausibility trace values (ρ) are determined.

(18) $\rho(A_k) = \sum_{i=1}^{k} m(H_i)$

(19) $\rho(A_1) = 0.370$

(20) $\rho(A_2) = 0.576$

(21) $\rho(A_3) = 0.788$

(22) $\rho(A_4) = 1.00$

Recall, that the plausibility trace becomes a possibility distribution when *A* is consistent. This example is repeated in Chapter 5 where the results of Options 1 and 2 are compared using Hamming distance, and Similarity.

3.3.4 Encounter with a Consonant Interval Set

The following examples are already consonant and do not need to be transformed. Their possibility histograms and thus possibility distributions are trivially obtained. The software recognizes this and only determines the possibility distribution.

Example 3.3

Given the following nested measurement record (columns 1 and 2), the resulting possibility distribution values are readily determined by the software (column 3).

Table 3.3a

Interval	Weight	Possibility Dist. Value
[5, 13)	1/3	1/3
[7, 13)	1/3	2/3
[9, 12)	1/3	1

In this trivial example, the interval set that comprises the core has the highest value in the possibility distribution, followed by the interval set that includes that interval and finally the least possible is the interval set that makes up the support.

Example 3.4

Given the observation record (columns 1 and 2), the possibility values are readily determined by the software (column 3).

Table 3.3b

Interval	Weight	Possibility Values
[36, 48)	1/4	1
[36, 48)	1/4	1
[36, 48)	1/4	1
[36, 48)	1/4	1

Notice that all the intervals are identical and thus all have the same possibility value.

3.4 Chapter Summary

All possibility histograms are fuzzy intervals and are described as possibility distributions from consistent empirical random sets [7]; they are derived from overlapping interval observations and are governed by the mathematics of random sets. Software has been developed using C++ to obtain plausibility trace values for various random interval sets. The use of software allows this study to take place more efficiently. The software sends these determined values to a data file, and using this data file the plausibility trace values are plotted to form possibility histograms. See the appendix for software. The methodology and utility of the transformation algorithms is demonstrated in this chapter through some illustrative examples.

Chapter 4

Analytical Method

4.1 Chapter Introduction

This chapter describes the metrics used in studying the transformed sets. The similarity and Hamming distance metrics are used to determine the closeness of the transformed set to the original set. The minimization and maximization of the Hamming distance and similarity, respectively, is our desired goal in the development of automated methods to select possibility distributions from a collection of random intervals. It is expected that the conflict among evidence (Strife) will be reduced through transformation but it is not clear how the nonspecificity among evidence will change. For example, there is less Strife in a consonant interval set than a non-consonant interval set thus the Strife should decrease through consonant transformation. The uncertainty measures used in this investigation are repeated in this chapter for clarity.

4.2 "Closeness"

The similarity and Hamming distance metrics are used to determine the "closeness" of the transformed set to the original set. The minimization and maximization of Hamming distance and similarity, respectively is desired in this investigation.

4.2.1 Hamming Distance Metric

Recall from Chapter 2, that we are dealing with a discrete set and not a continuous set. Therefore, the transformed set $(\widetilde{B})$ and the original set $(\widetilde{A})$ posses the same discrete observation values. The Hamming distance is readily calculated based on the following equation.

$$f(A) = \sum_{x \in X} \left| \rho_{\underset{\sim}{A}}(x) - \rho_{\underset{\sim}{B}}(x) \right|. \tag{4.2.1}$$

Demonstrating this for only the single discrete value 25, the distance for this single point is

$$f(25) = \left| \rho_{\widetilde{A}}(25) - \rho_{\widetilde{B}}(25) \right|.$$

At this discrete value, the original untransformed set has a plausibility trace value of 0.5 and in the transformed set the value is 0.6, the distance is thus equal to 0.1 for this single point.

$$f(25) = \left| (0.5) - (0.6) \right| = 0.1$$

This process is continued for all the discrete values defined over the support and summed to determine the overall Hamming Distance.

4.2.2 Similarity Metric

To determine the similarity value between an original untransformed set and a transformed set composed of three points 25, 26, and 27. The original set contains plausibility trace values of 0.5, 0.7, and 0.9 at 25, 26, and 27 respectively. The transformed set contains plausibility trace values of 0.6, 0.8, and 1.0 at 25, 26, and 27 respectively. To calculate the inner product, we first determine the minimum plausibility trace value at each point, next we determine the maximum of the resulting values. This is done using the following equation.

$$\underset{\sim}{A} \bullet \underset{\sim}{B}^{T} = \max_{i=1}^{n}(\min(\rho_{\tilde{A}}(x_i), \rho_{\tilde{B}}(x_i))). \tag{4.2.2}$$

$$\underset{\sim}{A} \bullet \underset{\sim}{B}^{T} = \max(\min(0.5,\ 0.6), \min(0.7, 0.8), \min(0.9, 1.0)) = 0.9$$

For the outer product, we utilize Equation 4.3 and determine the maximum of the complement of the plausibility values at points 25, 26, and 27, next we determine the minimum of the resulting values.

$$\overline{\underset{\sim}{A}} \oplus \overline{\underset{\sim}{B}}^{T} = \min_{i=1}^{n}(\max(\rho_{\overline{\tilde{A}}}(x_i), \rho_{\overline{\tilde{B}}}(x_i))). \tag{4.2.3}$$

$$\overline{\underset{\sim}{A}} \oplus \overline{\underset{\sim}{B}}^{T} = \min(\max(0.5, 0.4), \max(0.3, 0.2), \max(0.1, 0)) = 0.1$$

The outer product is thus equal to 0.1. Thus the similarity value is equal to the average of the inner and outer product.

$$(\underset{\sim}{A}, \underset{\sim}{B})_2 = \frac{1}{2}\left[(\underset{\sim}{A} \bullet \underset{\sim}{B}) + (\overline{\underset{\sim}{A}} \oplus \overline{\underset{\sim}{B}}^{T})\right] \tag{4.2.4}$$

$$(\underset{\sim}{A}, \underset{\sim}{B})_2 = \frac{1}{2}[0.9 + 0.1] = 0.5$$

4.3 Conflict

As mentioned in Chapter 2, several measures for conflict among evidence have been proposed. Recall, that conflict is the disagreement among the random intervals of the random set. For this study, the most valuable measure of conflict among evidence is found using the equation for Strife (ST), which is defined by the formula below.

$$\mathrm{ST(m)} = -\sum_{A_1 \in S_R} \mathrm{m}(A_1) \log_2 \sum_{A_2 \in S_R} \mathrm{m(B)} \frac{|A_1 \cap A_2|}{|A_1|} \tag{4.3.1}$$

where, A_1 and A_2 are focal sets of A and $\mathrm{m}(\cdot)$ is a measure of the weight. The above formula requires that the number of focal elements in A_1 and A_2 be known, that we have a discrete interval set. This allows the cardinality of a set, say $|A_1|$, to be determined. In Chapter 2, it was stated that discrete rather than continuous sets are used and that the granularity of these discrete sets is 1. Strife properly scales each particular conflict of m(B) with respect to m(A) according to the degree of violation of the subsethood relation $B \subseteq A$ [6].

4.3.1 Strife

Consider the Example 3.3 of Chapter 3, Strife (ST) is calculated using Equation 4.3.1.

$$\mathrm{ST(m)} = \left(-\frac{1}{3}\right) * \log_2 \left[\frac{1}{3} * \frac{|[5,13) \cap [7,13)|}{|[5,13)|} + \frac{1}{3} * \frac{|[5,13) \cap [9,12)|}{|[5,13)|}\right] +$$

$$\left(-\frac{1}{3}\right) * \log_2 \left[\frac{1}{3} * \frac{|[7,13) \cap [5,13)|}{|[7,13)|} + \frac{1}{3} * \frac{|[7,13) \cap [9,12)|}{|[7,13)|}\right] +$$

$$\left(-\frac{1}{3}\right)*\log_2\left[\frac{1}{3}*\frac{|[9,12)\cap[5,13)|}{|[9,12)|}+\frac{1}{3}*\frac{|[9,12)\cap[7,13)|}{|[9,12)|}\right]$$

$$\mathrm{ST(m)}=\left(-\frac{1}{3}\right)*\left(\log_2\left[\frac{1}{3}*\frac{6}{8}+\frac{1}{3}*\frac{3}{8}\right]+\log_2\left[\frac{1}{3}*\frac{6}{6}+\frac{1}{3}*\frac{3}{6}\right]+\log_2\left[\frac{1}{3}*\frac{3}{3}+\frac{1}{3}*\frac{3}{3}\right]\right)$$

$$\mathrm{ST(m)}=0.9999\overline{9}\approx 1$$

This value is useful in comparing sets of evidence, the lower the value of Strife the less the conflict among the evidence of the set.

4.4 Nonspecificity

As stated in Chapter 2, nonspecificity is related to sizes of the relevant sets of alternatives and for evidence theory Nonspecificity (N) is reasonable measure of uncertainty [6]. For a discrete set, the value of N will be larger for sets with more members of elements. Thus making the random set less specific. Either the function for U-uncertainty (U) or Nonspecificity (N) may be applied in this investigation. Recall, U is applicable in fuzzy set theory and possibility theory, N is applicable in evidence theory, and Hartley function is applicable to probability theory [6]. For nested bodies of evidence, U is a special case of N [6], and is equal to N, otherwise $\mathrm{U} \neq \mathrm{N}$. A normal possibility histogram is determined for each transformed set of interval data, be it consonant or only consistent. U can be calculated for each of these resulting histograms (fuzzy intervals F) using the Equation 4.4.1; however, the value obtained may differ from that of N.

$$\mathrm{U}(F)=\int_0^1 \log_2 |{}^{\pi}F|\, \mathrm{d}\pi \qquad (4.4.1)$$

where $d\pi$ indicates the degree of possibility focusing on $|^{\pi}F|$ and F indicates a fuzzy interval.

While U is obtained from the resulting possibility distribution, N is determined using the weight m(Ai), of each focal element A_i of A and the cardinality of A_i.

$$N(m) = \sum_{A_i \in S_R} m(A_i) \log_2 |A_i| \tag{4.4.2}$$

The larger the value of A_i, the less specific the evidence; and the larger the $m(A_i)$, the stronger the evidence [6].

4.4.1 Function U-uncertainty (U)

Again considering the Example 3.3 and figure 3.3a of Chapter 3, U-uncertainty is calculated using Eq. 4.4.1 as follows (in units of bits).

$$U(A) = \int_0^{1/3} \log_2 |8| \, d\pi + \int_{1/3}^{2/3} \log_2 |6| \, d\pi + \int_{2/3}^{1} \log_2 |3| \, d\pi = 2.389$$

4.4.2 Function Nonspecificity (N)

Continuing with Example 3.3 Chapter 3, Nonspecificity (N) is calculated using Equation 4.4.2 as follows.

$$N(m) = \tfrac{1}{3} \log_2 (13-5) + \tfrac{1}{3} \log_2 (13-7) + \tfrac{1}{3} \log_2 (12-9) = 2.39 \text{ bits}$$

Notice that the evidence of Example 3.3 is nested thus U = N.

4.5 Aggregate Uncertainty

A metric has been proposed that is supposed to capture, in an aggregate fashion, both the nonspecificity and the conflict in a given body evidence (expressed in terms of evidence theory); this metric is known as aggregate uncertainty (AU) and can be expressed using belief measures, plausibility measures, or basic probability assignments [6]. Two proposed algorithms were presented in Chapter 2 for calculating AU, (1) using belief measures or (2) using the possibility measures from a nested set of interval data.
Both 1 and 2 are used in this investigation and are illustrated in sections 4.5.1 and 4.5.2.

4.5.1 Aggregate Uncertainty Using Possibility Measures

Again, continuing with Example 3.3 and the Algorithm 2.6.1 for AU introduced in Chapter 2, for nested evidence and using the possibility distribution values for the elements of the discrete interval sets (π_i), AU is determined as follows.

Table 4.5a

Element	Possibility
$x_1 = 9$	1
$x_2 = 10$	1
$x_3 = 11$	1
$x_4 = 7$	2/3
$x_5 = 8$	2/3
$x_6 = 12$	2/3
$x_7 = 5$	1/3
$x_8 = 6$	1/3

Step 1. Let j = 1 and $\pi_{n+1} = 0$, where n = 8

Step 2. Find the maximum $i \in \{j, j+1, \ldots, n\}$ such that $\frac{\pi_j - \pi_{i+1}}{i+1-j}$ is maximal.

$$\frac{\pi_1 - \pi_2}{1+1-1} = \frac{1-1}{1} = 0$$

$$\frac{\pi_1 - \pi_4}{3+1-1} = \frac{1-.666\overline{6}}{3} = 0.1112$$

$$\frac{\pi_1 - \pi_6}{6+1-1} = \frac{1-.333\overline{3}}{6} = 0.1112$$

$$\frac{\pi_1 - \pi_9}{8+1-1} = \frac{1-0}{8} = 0.125 \text{ *Maximal}$$

Step 3. For $k\{1,2,3,\ldots 8\}$ put $p_k = \frac{\pi_1 - \pi_9}{8+1-1} = \frac{1-0}{8} = 0.125$

Step 4. j = 8 + 1

Step 5. Since i > n, go to Step 6.

Step 6. $AU(\Pi) = -8 * 0.125 \log_2(0.125) = 3$ bits

The Aggregate Uncertainty is 3 bits for Example 3.3. This value is combines the uncertainty in evidence due to Strife and Nonspecificity in an aggregate manner and is used to compare the uncertainty in transformed sets.

4.5.2 Aggregate Uncertainty Using Belief Measures

Continuing with Example 3.3 and Algorithm 2.6.2 for AU introduced in Chapter 2 using the belief measures, AU is determined as follows, X = {5,6,7,…,12}

Table 4.5b

A	Bel(A)	$\frac{Bel(A)}{\lvert A \rvert}$
{9,10,11}	1/3	0.1112
{7,8,9,10,11,12}	2/3	0.1112
{5,6,7,8,9,10,11,12}	1	0.125*

Step 1. * is Maximal

Step 2. For $x \in A$, put $p_x = 0.125$

Step 3. For each $B \subseteq [X - A]$, put $Bel(B) = Bel(B \cup A) - Bel(A)$.

Since $X - A = 0$, skip to Step 7.

Step 7. $AU(\Pi) = -8 * 0.125 \log_2(0.125) = 3$ bits

The set in the above examples was consonant (i.e. nested), thus the AU obtained through Algorithm 2.6.1 and 2.6.2 are equivalent. Algorithm 2.6.1 and 2.6.2 are both applicable to consonant interval sets however, only Algorithm 2.6.2 is applicable to consistent but non-consonant interval sets.

4.6 Chapter Summary

Conflict in evidence among a collection of random intervals is measured through the use of a function called Strife(ST). Likewise, the nonspecificity is measured through the use of the function called Nonspecificity (N). Both conflict and nonspecificity in evidence are captured by the Aggregate Uncertainty (AU). For fuzzy sets, the function U can be utilized to calculate the nonspecificity. The use of the possibility distribution for the determination of AU is only applicable to discrete consonant interval sets whereas, the use of belief is applicable to all types of discrete interval sets. For nested set of evidence U is a special case of N, where N = U.

Chapter 5

Comparison of Transformed Sets

5.1 Chapter Introduction

In transforming an inconsistent random set to a consistent set, there exists an array of possible transformed consistent sets depending on the normalizing interval chosen. The same is true for the transformation of non-consonant random sets. This array of possible transformed sets is a consequence of various normalizing intervals available for consistent transformation and the various collections of consonant intervals available for consonant transformation. The transformed sets included in this array differ from one another; this is apparent, not only in the resulting possibility histogram of each transformed set, but in the conflict and nonspecificity measures of each transformed set. The resulting possibility histograms, Nonspecificity, Strife, and Aggregate Uncertainty can all be used to reveal the effects of certain characteristics of the normalizing interval or intervals on consistent and consonant transformations, respectively. Similarity and Hamming distance values are calculated to determine the "likeness" of the transformed set to that of the original set. Recall, these metrics are based on the plausibility trace of the untransformed sets and the transformed sets. The Hamming distance, Similarity, Strife, Nonspecificity and Aggregate Uncertainty values for the array of transformed sets are displayed in the last table for each transformation example of this chapter.

5.2 Examples

This section introduces 7 interval set examples transformable using consonant transformation (Donald algorithm) or consistent transformation (Joslyn algorithm). The examples consist of non-consistent sets to consistent but non-consonant sets.

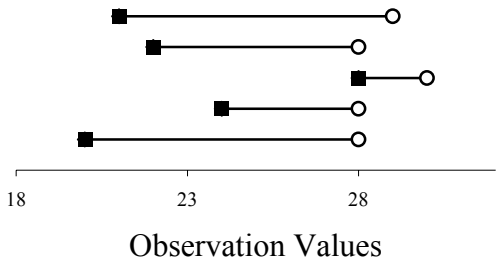
Example I

This example was introduced in Chapter 3 as Example 3.2. It is a non-consistent but overlapping interval set (see Figure 5.1), the observational interval set is repeated here. Due to the nature of the observation set, overlapping, it may be transformed using either the transformation algorithm proposed by Jolsyn [8, 9] or Donald [4]. Only consonant transformation proposed by Donald [4] is pursued here (see Section 5.3)

Table 5.1

Interval	Weight
[20, 28)	0.2
[24, 28)	0.2
[28, 30)	0.2
[22, 28)	0.2
[21, 29)	0.2

Figure 5.1



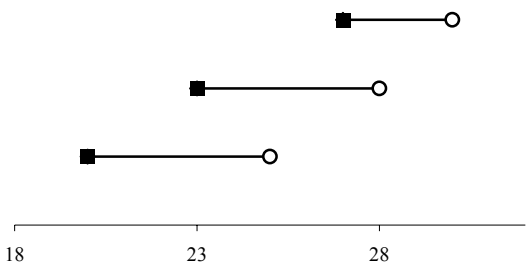
Example II

This example is also a non-consistent but overlapping interval set, again both transformations algorithms are applicable. Both transformations are demonstrated in this chapter.

Table 5.2

Observational Interval	Weight
[20, 25)	1/3
[23, 28)	1/3
[27, 30)	1/3

Figure 5.2



Observation Values

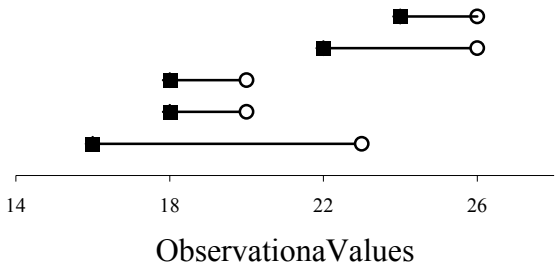
Example III

This is another example of an overlapping set however, it is important to note that interval [18, 20) occurs twice in this interval set.

Table 5.3

Observational Interval	Weight
[16, 23)	1/5
[18, 20)	1/5
[18, 20)	1/5
[22, 26)	1/5
[24, 26)	1/5

Figure 5.3



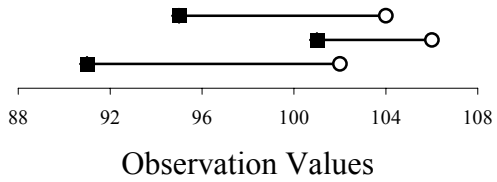
Example IV

This is a consistent but non-consistent interval set, with its core equal to one width. It is important to note that only consonant transformation is applicable here, already consistent.

Table 5.4

Observational Interval	Weight
[91, 102)	1/3
[101, 106)	1/3
[95, 104)	1/3

Figure 5.4



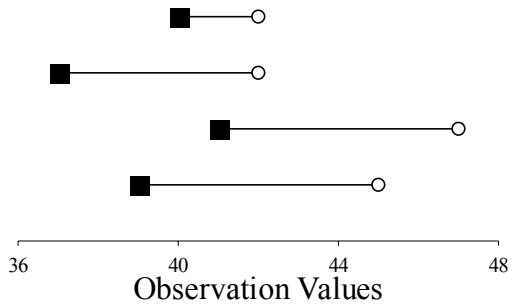
Example V

The interval set here is non-consonant but consistent, again note that it may be transformed to a consonant set only (being that it is already consistent).

Table 5.5

Observational Interval	Weight
[39, 45)	1/4
[41, 47)	1/4
[37, 42)	1/4
[40, 42)	1/4

Figure 5.5



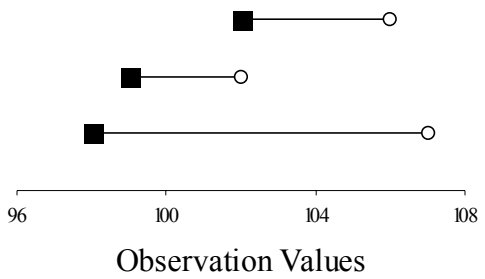
Example VI

The following interval set is non-consistent but overlapping. An important characteristic is that this interval set contains a super-interval that is equivalent to the support and includes both sub-intervals.

Table 5.6

Observational Interval	Weight
[98, 107)	1/3
[99, 102)	1/3
[102, 106)	1/3

Figure 5.6



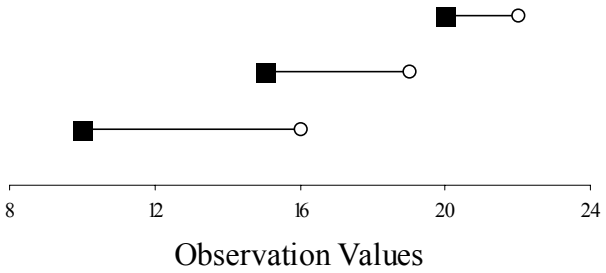
Example VII

This interval set here is a disjoint interval set, no evidence is available over a certain portion of the support. It is may transformed consistent (Jolsyn algorithm) or consonant (Donald algorithm), however it only transformed to a consistent interval set in this chapter (see Section 5.4).

Table 5.7

Observational Interval	Weight
[10, 16)	1/3
[15, 19)	1/3
[20, 22)	1/3

Figure 5.7



5.3 Consonant Transformation

In consonant transformation, a non-consonant interval set is transformed to a consonant interval set. Recall that for consonant interval sets U becomes N, thus U is not included in the following tables. In the following examples, the algorithm proposed by Donald & Ross [4] is used to transform the non-consonant set to a consonant set. As mentioned in Chapter 1, various transformed consonant interval sets may result from a non-consonant interval set; this is due to the various collections of selected normalizing intervals from the focal set in transforming the non-consonant set. Each collection of normalizing intervals produces a fairly different transformed set. Each resulting transformed interval set is denoted as simply an Option. In Example 5.1, two transformed consonant interval options are considered. For each transformed option, the Hamming distance, Similarity, Strife, Nonspecificity, and Aggregate Uncertainty are calculated. This allows the comparison between the transformed sets. The Hamming distance and Similarity metrics use the plausibility trace ρ of both the original set and the transformed set in calculating their resulting values. Recall that the plausibility trace is a mapping of the plausibility of the singleton elements, x of the random set A to the unit interval.

Example 5.1 (refer to Example I above)

Let's begin with the non-consistent Example I of Section 5.2 above, the resulting possibility distribution for two options (Option 1A and Option 1B) are illustrated in Figures 5.1a and 5.1b. Included in these figures are the plausibility trace for the original interval set. Strife (ST), Nonspecificity (N), and Aggregate Uncertainty (AU) for both

Option 1A and 1B are shown in Table 5.1a. Also shown in Table 5.1a are the Similarity and Hamming distance metrics.

Figure 5.1a

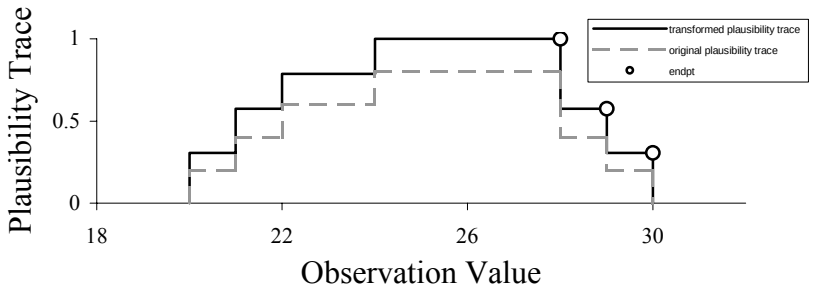


Figure 5.1b

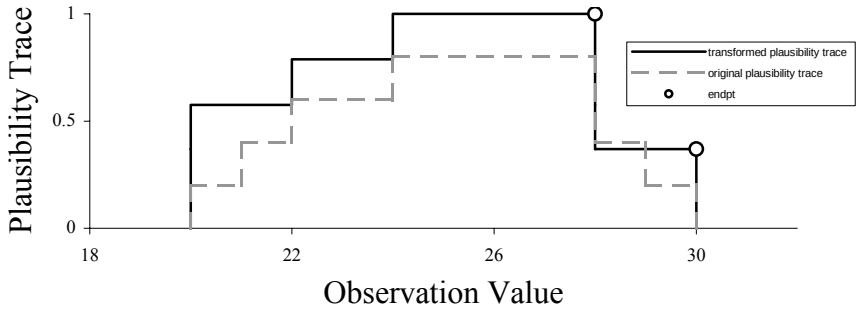


Table 5.1a

	Nonspecificity	Strife	Aggregate	Similarity	Distance
Option 1A	2.798	0.767	3.322	**0.747**	**1.739**
Option 1B	2.819	0.830	3.322	0.715	2.128

Option 1A has a higher Similarity as well as lower Hamming distance value, hence it is more like that of the original set. The normalizing intervals of Option 1A and Option 1B are very similar, they differ in only one interval [20, 28) and [21, 29) (see Example 3.2 of Chapter 3). These two intervals are included by the same number of super-intervals and have the same plausibility; however, the number of elements in common with the rest of

the focal set is what sets them apart. The interval [21, 29), of Option 1A, has more points in common with the focal set than does interval [20, 28).

Example 5.2 (refer to Example II above)

The following intervals of Table 5.2a are formed from the intersections of the above intervals and form the focal set (see Section 2.5.1 of Chapter 2). Notice that the min-max interval (equivalent to the support) has been included in the focal set, this is from the union of all sets and is included in the focal set if it has not been accounted for in the original set. The determination of the original weight and the normalized weight is explained in Chapter 2 and is not repeated here.

Table 5.2a

Focal Set	Original Weight	Normalized Weight
[20, 30)	1/3	1/6
[20, 25)	1/3	1/6
[23, 25)	1/3	1/6
[27, 28)	1/3	1/6
[23, 28)	1/3	1/6
[27, 30)	1/3	1/6

There are four apparent options (2A, 2B, 2C, 2D), for transforming the above non-consonant set to a consonant set. Note that the support is included in each of the Options.

Table 5.2b

Option 2A		Option 2B		Option 2C		Option 2D	
H	I	H	I	H	I	H	I
[20, 30)	[23, 28)	[20, 30)	[20, 25)	[20, 30)	[20, 25)	[20, 30)	[20,25)
[20, 25)	[27, 30)	[23, 28)	[23, 25)	[27, 30)	[23, 25)	[23, 28)	[27,28)
[23, 25)	[27, 28)	[27, 28)	[27, 30)	[27, 28)	[23, 28)	[23, 25)	[27,30)

The resulting plausibility trace of Options 2A, 2B, 2C and 2D are given in Figures 5.2a, 5.2b, 5.2c and 5.2d, respectively.

Figure 5.2a

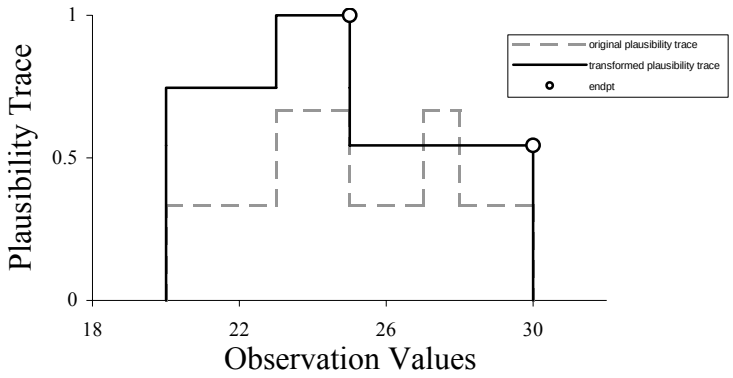


Figure 5.2b

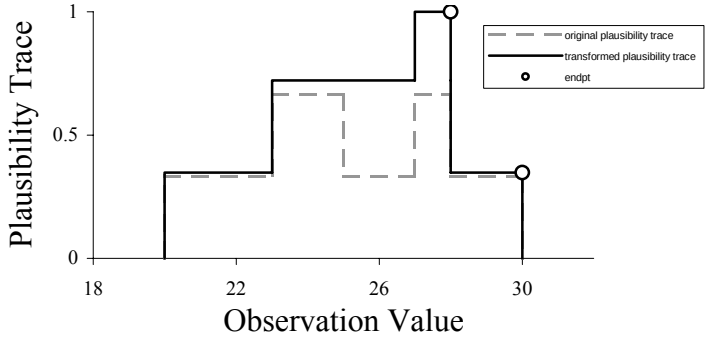


Figure 5.2c

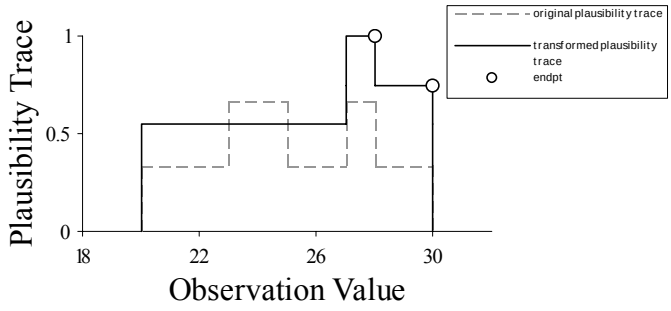


Figure 5.2d

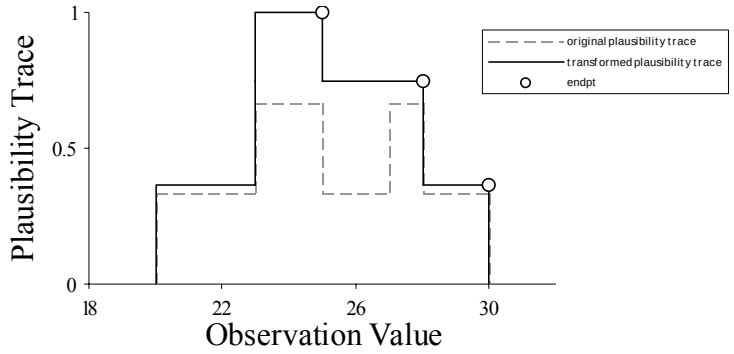


Table 5.2c

	N	ST	AU	Similarity	Distance
Option 2A	2.529	1.714	3.309	0.561	3.0263
Option 2B	2.025	1.392	3.130	0.659	1.29629
Option 2C	2.124	2.183	3.170	0.561	2.4545
Option 2D	2.352	1.276	3.269	0.650	1.7339

As stated above, the support is included in each of the Options; however, the selection of the following intervals which result in a transformed set that is most like that of the original set is desired. Each of these non-minimal intervals is included by an equivalent amount of super-intervals (the support), however they differ in their plausibility as well as the number of elements in common. The minimal intervals differ in size but have an equivalent plausibility. In this example, Option 2B produces the transformed set that is most like that of the original set (as determined from Similarity and Hamming distance). The minimal interval of Option 2B is the smallest. The non-minimal interval [23, 28) has the highest plausibility and the most points in common with the rest of the focal set.

Example 5.3 (refer to Example III above)

Consider yet another non-consonant and inconsistent but overlapping observation record as shown in Table 5.3. Here there are two apparent alternatives for consonant transformation without determining any additional intervals as was done in Example 5.1. However, if the approach taken in Example 5.2 is taken here there will be 2 additional alternatives. Table 5.3a includes the intervals formed from the intersection of the above intervals. The original weight is based on the minimum norm and the assumption that the underlying evidence is coherent (see Section 2.5.1 of Chapter 2). The resulting 4 consonant alternatives are shown Table 5.3b and in Figures 5.3a, 5.3b, 5.3c, and 5.3d.

Table 5.3a

Focal Set	Original Weight	Normalized Weight
[16, 26)	1/5	1/7
[16, 23)	1/5	1/7
[18, 20)	2/5	2/7
[22, 23)	1/5	1/7
[22, 26)	1/5	1/7
[24, 26)	1/5	1/7

Table 5.3b

Option 3A		Option 3B		Option 3C		Option 3D	
H	I	H	I	H	I	H	I
[16, 26)	[22, 23)	[16, 26)	[18, 20)	[16, 26)	[16, 23)	[16, 26)	[16, 23)
[16, 23)	[22, 26)	[16, 23)	[18, 20)	[22, 26)	[18, 20)	[22, 26)	[18, 20)
[18, 20)	[24, 26)	[22, 23)	[22, 26)	[22, 23)	[18, 20)	[24, 26)	[18, 20)
[18, 20)			[24, 26]		[24, 26)		

Figure 5.3a

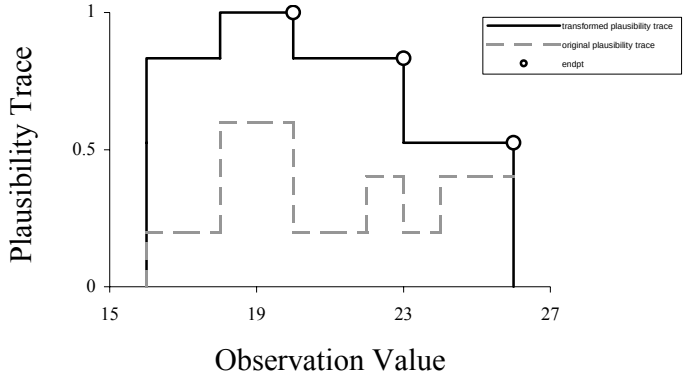


Figure 5.3b

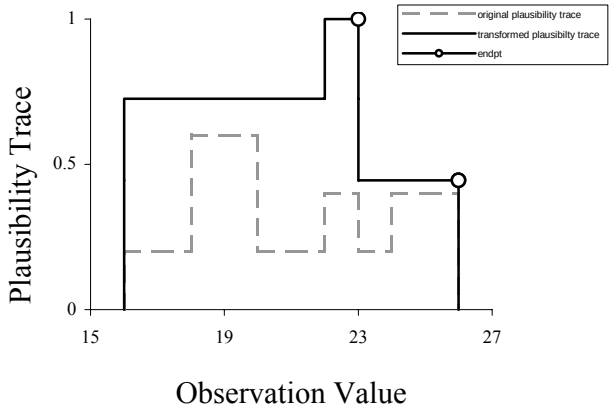


Figure 5.3c

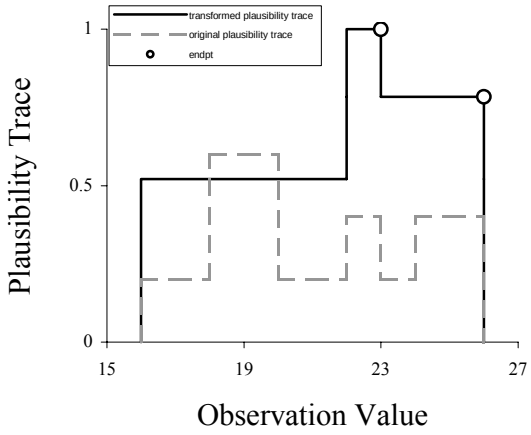


Figure 5.3d

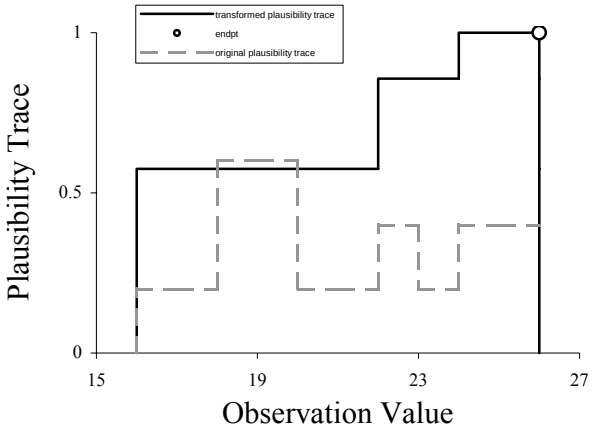


Table 5.3c

	N	ST	AU	Similarity	Distance
Option 3A	2.781	1.203	3.322	0.575	4.235
Option 3B	2.373	1.320	3.211	**0.580**	**3.508**
Option 3C	2.256	1.837	3.238	0.500	3.393
Option 3D	2.616	1.831	3.309	0.500	3.721

Again in this example, non-minimal intervals with the maximum plausibility result in a transformed set that is most like that of the original set, according to Similarity and Hamming distance values. Option 3A and Option 3B both contain intervals [16, 26) and [16, 23), however they differ in the following interval (i.e. [18, 20) and [22, 23), respectively). In this case, both interval [22, 23) and interval [18, 20) are minimal but [22, 23) is smaller; however, they [18, 20) has a higher plausibility trace. Option 3B is most like the original set; this is demonstrated by a maximal value of Similarity and minimal value of Hamming Distance.

From the results, it seems that selecting the non-minimal interval over the minimal interval will result in a transformed set that is most like that of the original interval set.

Example 5.4 (refer to Example IV above)

Following the same procedure as was done above in Example 5.2, the following intervals and weight are obtained.

Table 5.4a

Focal Set	Original Weight	Normalized Weight
[91, 106)	1/3	1/7
[91, 102)	1/3	1/7
[101, 102)	1/3	1/7
[101, 106)	1/3	1/7
[101, 104)	1/3	1/7
[95, 102)	1/3	1/7
[95, 104)	1/3	1/7

Here there are three options for transforming the above non-consonant set to a consonant set.

Table 5.4b

Option 4A		Option 4B		Option 4C	
H	I	H	I	H	I
[91, 106)	[101, 106)	[91, 106)	[91, 102)	[91, 106)	[91, 102)
[91, 102)	[101, 104)	[101, 106)	[95, 102)	[95, 104)	[101, 106)
[95, 102)	[95, 104)	[101, 104)	[95, 104)	[95, 102)	[101, 104)
[101, 102)		[101, 102)		[101, 102)	

Option 4D	
H	I
[91, 106)	[91, 102)
[95, 104)	[95, 102)
[101, 104)	[101, 106)

[101, 102)	

The resulting possibility distribution for Options 4A, 4B, 4C and 4D are shown in Figures 5.4a, 5.4b, 5.4c, and 5.4d respectively.

Figure 5.4a

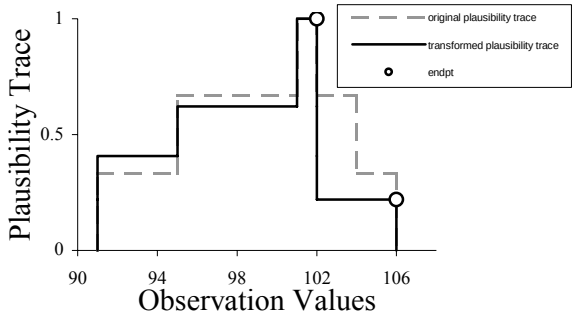


Figure 5.4b

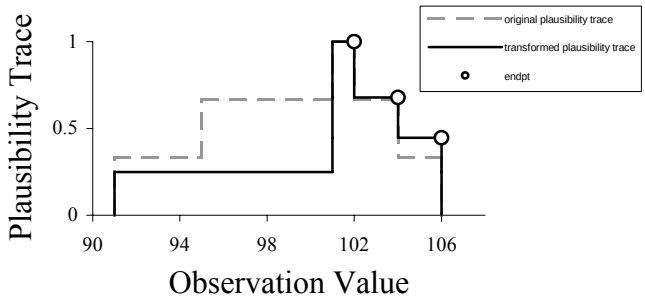


Figure 5.4c

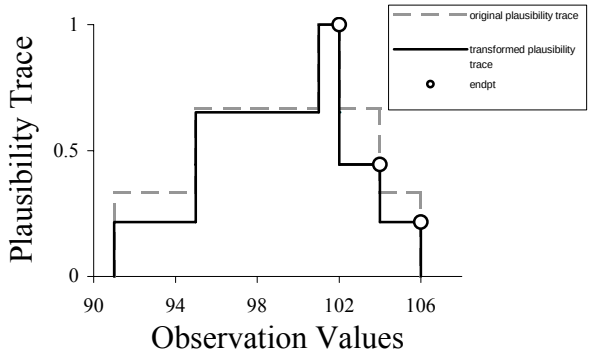


Figure 5.4d

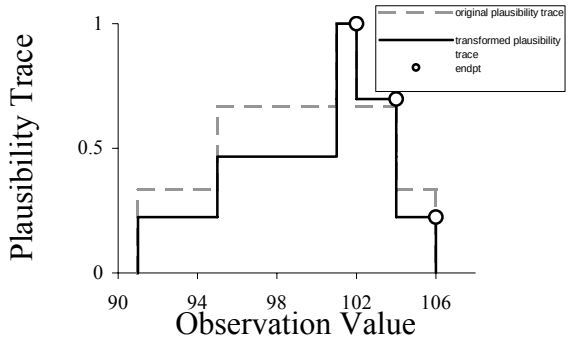


Table 5.4c

	N	ST	AU	Similarity	Distance
Option 4A	2.111	1.292	3.325	0.796	1.687
Option 4B	1.798	1.328	3.116	0.833	3.089
Option 4C	2.152	1.138	3.397	0.833	1.026
Option 4D	2.010	1.230	3.397	0.651	2.034

Here all the Options have the same minimal interval in common ([101, 102)); therefore, the difference is among the non-minimal intervals. Each of the intervals in Table 5.4a have an equivalent plausibility, however they differ in points in common with the rest of the focal set. Based on maximal Similarity and minimal Hamming distance, Option 4C is most like that of the original set. Interval [95, 104) has more points in common with the rest of the focal set than does intervals [91, 102) and [101, 106). Also, interval [95, 102) has more points in common than interval [101, 104).

Example 5.5

It is important to note that this is a consistent but non-consonant interval set. The non-consonant set can be transformed consonant by the following options.

Table 5.5a

Option 5A		Option 5B		Option 5C	
H_i	I_j	H_i	I_j	H_j	I_j
[37, 47)	[41, 47)	[37, 47)	[39, 45)	[37, 47)	[39, 45)
[39, 45)	[37, 42)	[37, 42)	[41, 47)	[41, 47)	[37, 42)
[40, 42)		[40, 42)			[40, 42)

Which results in the following transformed sets.

Figure 5.5a

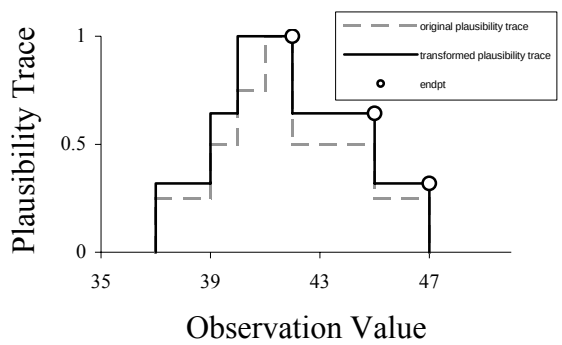


Figure 5.5b

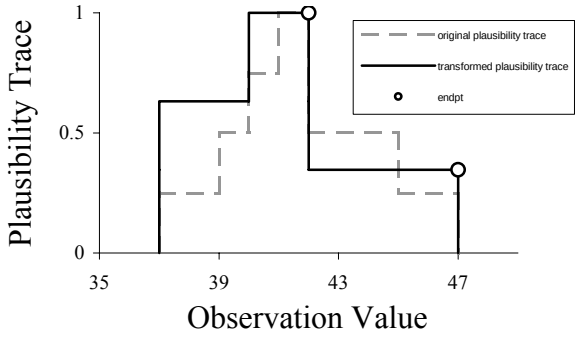
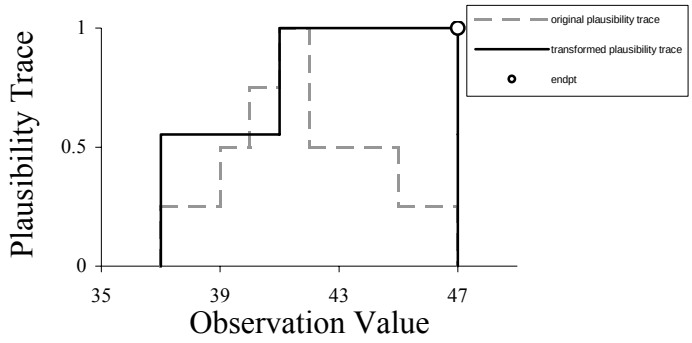


Figure 5.5c



However, the original observation record can be further broken down as was done in Example 5.2 resulting in the following intervals and weight are obtained.

Table 5.5b

Focal Set	Original Weight	Normalized Weight
[37, 47)	1/4	1/8
[39, 45)	1/4	1/8
[41, 45)	1/4	1/8
[39, 42)	1/4	1/8
[40, 42)	1/4	1/8
[41, 47)	1/4	1/8
[41, 42)	1/4	1/8
[37, 42)	1/4	1/8

From which there are 4 available alternatives for transforming the non-consonant set to a consonant set.

Figure 5.5d

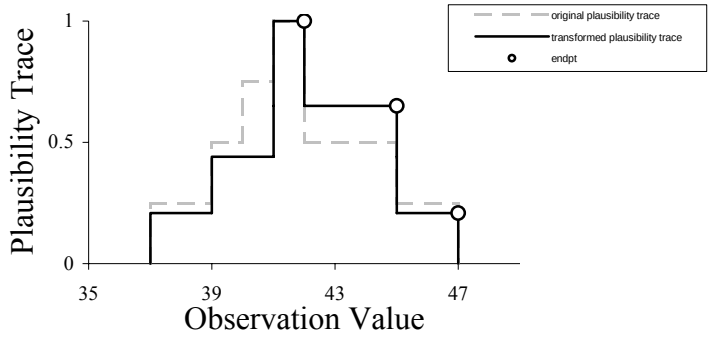


Figure 5.5e

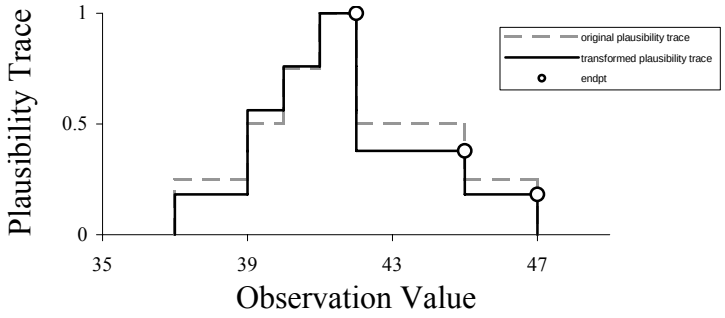


Figure 5.5f

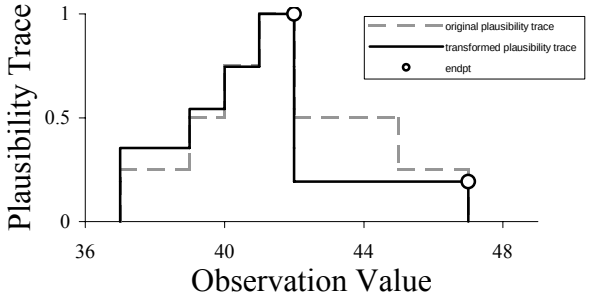


Figure 5.5g

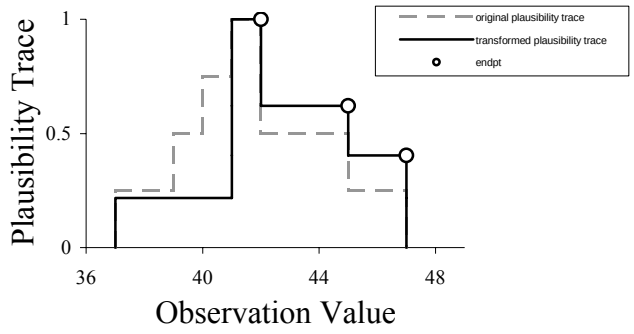


Table 5.5c

	N	ST	AU	Similarity	Distance
Option 5A	2.254	1.222	3.227	0.841	1.095
Option 5B	2.182	1.300	3.202	0.826	1.800
Option 5C	2.993	1.433	3.322	0.723	3.715
Option 5D	1.711	1.119	2.964	0.875	0.988
Option 5E	1.602	0.997	2.985	0.875	0.709
Option 5F	1.640	1.160	2.910	0.875	1.434
Option 5G	1.517	1.010	2.782	0.820245	1.295

Example 5.6

Given the non-consonant observation record of Table 5.6, there are only two available options for transforming it to a consonant set. Note that the interval set is non-consistent and non-consonant but one super-interval includes the other two sub-intervals. The possibility distributions for options 6A and 6B are shown in Figures 5.6a and 5.6b, respectively.

Table 5.6a

Option 6A		Option 6B	
H_i	I_j	H_i	I_j
[98, 107)	[102, 106)	[98, 107)	[99, 102)
[99, 102)		[102, 106)	

Figure 5.6a

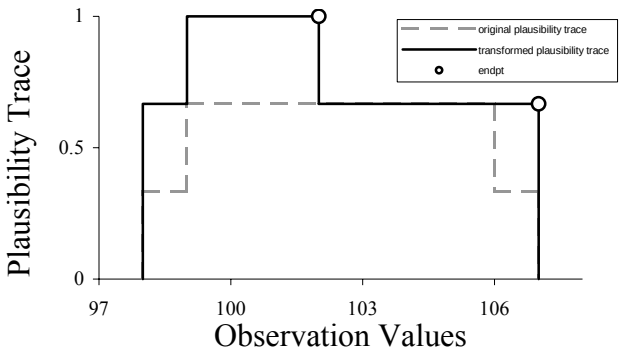


Figure 5.6b

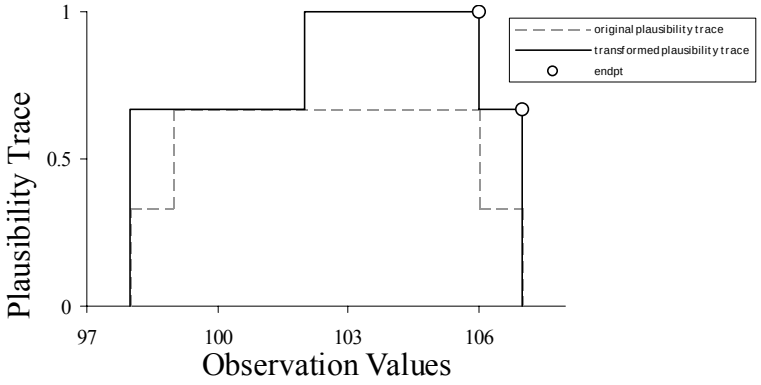


Table 5.6b

	N	ST	AU	Similarity	Distance
Option 6A	2.642	2.308	3.170	0.500	1
Option 6B	2.780	2.032	3.170	0.500	1.333

Both options are comprised of only two intervals, the super-interval being equivalent. The minimal intervals have an equivalent plausibility but are differ in size. Again, the option that uses the minimal interval with minimal size and maximal plausibility is most like the original set, based on maximal Similarity and minimal Hamming distance.

5.4 Consistent Transformation

In the following examples asterisk denotes a transformed consonant set and includes the measures N but not U. For transformed consistent but non-consonant random sets both U and N are included.

Example 5.7

Given the following inconsistent and disjoint observation record of Table 5.7 of Example VII (Section 5.2), it is transformed consistent using Joslyn's Algorithm. The normalizing intervals are limited to those that form the finest partition of the support $\Gamma = \{G_k\}$, (see Section 2.5.2) of which there are 5 and thus 5 alternative transformed consistent sets. Table 5.7a illustrates the available normalizing intervals. The plausibility trace for the resulting transformed sets for Options 7A, 7B, 7C, 7D and 7E are shown in Figures 5.7a, b, c, d,and e, respectively.

Table 5.7a

Option	G_k	Plausibility Trace
7A	[10, 15)	1/3
7B	[15, 16)	2/3
7C	[16, 19)	1/3
7D	[19, 20)	0
7E	[20, 22)	1/3

Figure 5.7a

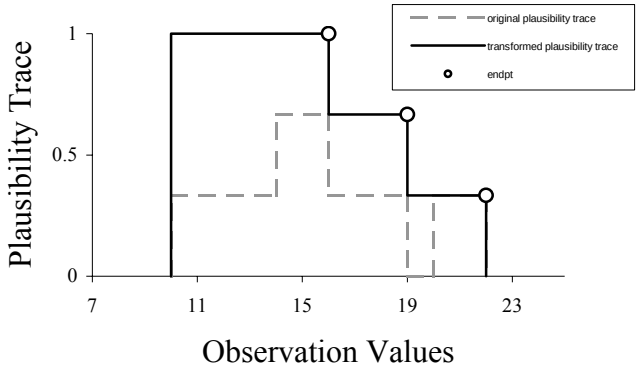


Figure 5.7b

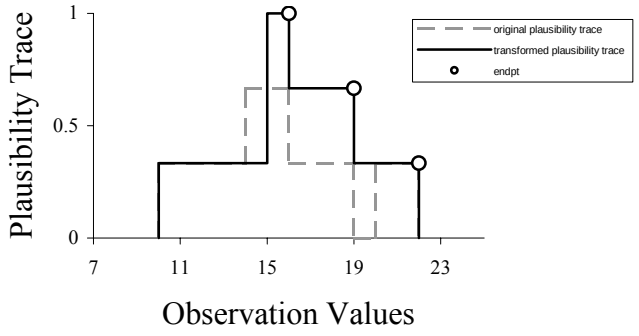


Figure 5.7c

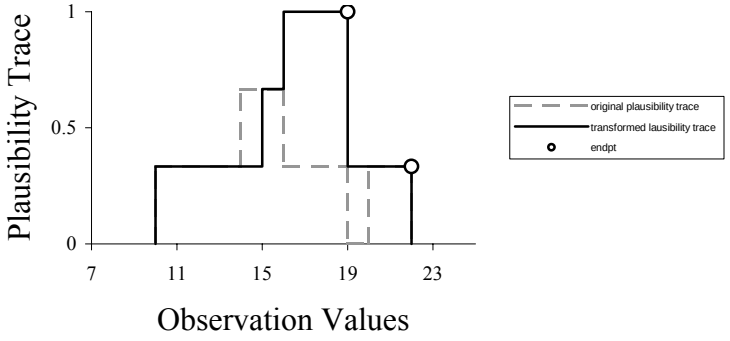


Figure 5.7d

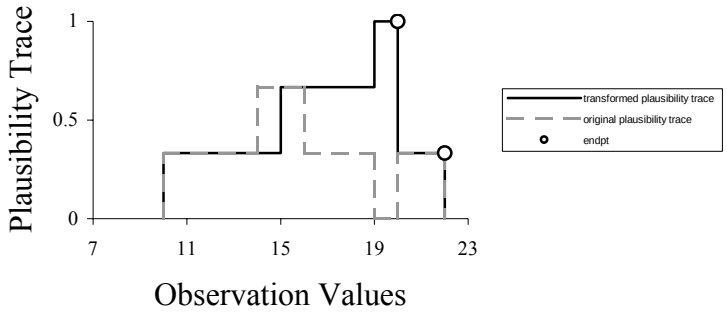


Figure 5.7e

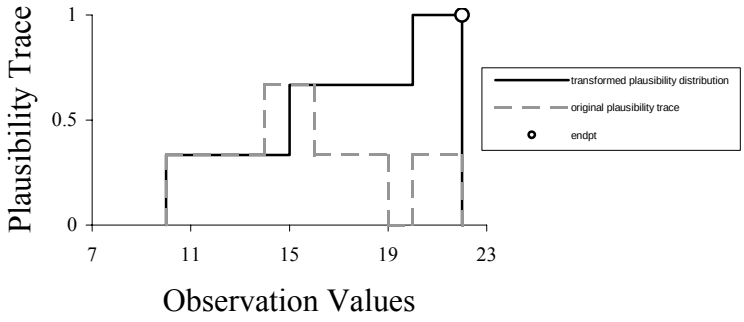


Table 5.7b

	N	U	ST	AU	Similarity	Distance
Option 7A*	3.113	3.113	0.8987	3.585	2/6	4.667
Option 7B	2.464	1.8616	2.168	3.563	2/6	**1.667**
Option 7C	2.585	2.195	1.437	3.585	2/6	2.000
Option 7D	2.410	1.969	1.938	3.560	2/6	2.000
Option 7E*	2.528	2.528	1.269	3.466	2/6	3.000

Notice that the consonant sets Options 7A and 7E are the farthest from the original set; also, that equivalent values for Similarity are produced. Here selecting the smallest normalizing interval with the highest plausibility results in the transformed set most like that of the original set. Based on minimal Hamming Distance, this set that is most like that of the original is Option 7B.

Example 5.8

Now consider consistent transformation of the non-consistent but overlapping set of Example II in Section 5.2. By limiting the normalizing intervals to those that form the finest partition of the support, we have 5 normalizing intervals and thus 5 transformed consistent alternatives. Table 5.8a illustrates the normalizing intervals used in transforming the original set. The possibility distribution for each Option A, B, C, D, and E are in Figures 5.8a, b, c, d, e, respectively.

Table 5.8a

Option	G_k	Plausibility Trace
8A	[20, 23)	1/3
8B	[23, 25)	2/3
8C	[25, 28)	1/3
8D	[27, 28)	2/3
8E	[28, 30)	1/3

Figure 5.8a

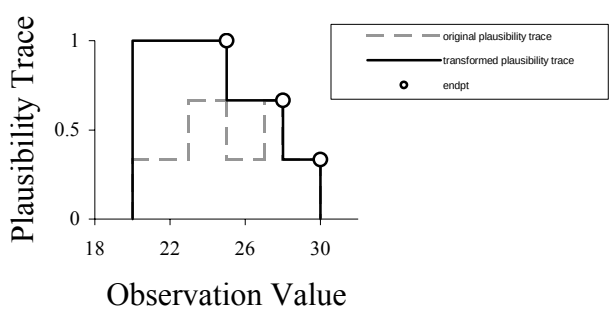


Figure 5.8b

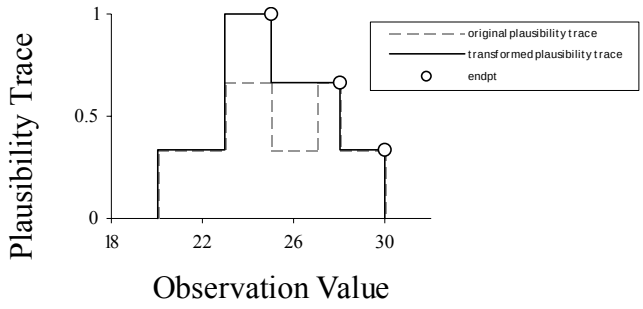


Figure 5.8c

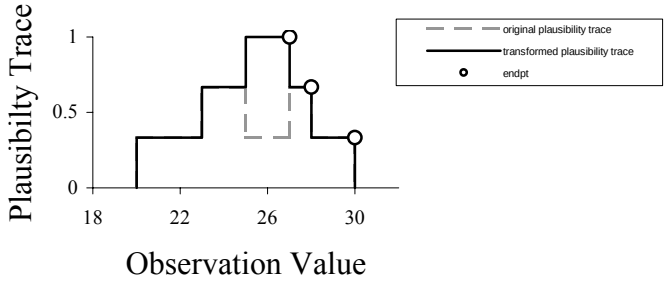


Figure 5.8d

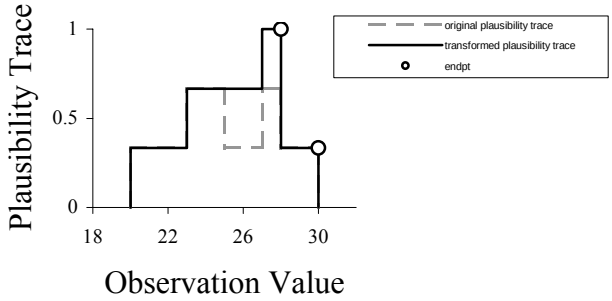


Figure 5.8e

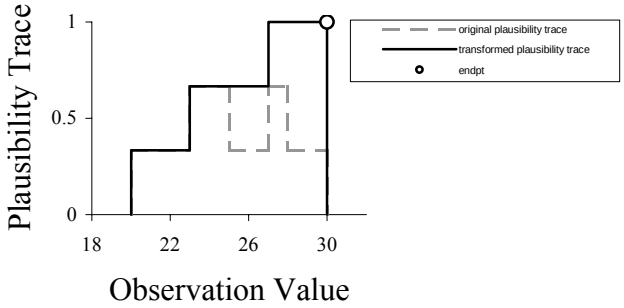


Table 5.8b

	N	U	ST	AU	Similarity	Distance
Option 8A*	2.881	2.881	0.892	3.322	2/3	3.334
Option 8B	2.484	2.215	1.530	3.322	2/3	1.334
Option 8C	2.484	2.215	1.497	3.322	2/3	1.334
Option 8D	2.302	1.881	1.831	3.319	2/3	**1.000**
Option 8E*	2.571	2.571	1.080	3.258	2/3	2.334

Again, selecting the smallest normalizing interval with the highest plausibility results in the lowest Hamming distance, which is the transformed consistent set that is most like that of the original set.

Example 5.9

Returning to Example III above (see Section 5.2), consisting of an inconsistent but overlapping observation interval. Again, note that the observation [18, 20) occurs twice in the observation interval. By limiting the normalizing intervals to those that form the finest partition of the support, there are 6 normalizing intervals (Table 5.9a) and thus 6 transformed consistent alternatives. The possibility distribution for each Option 9A, 9B, 9C, 9D, 9E, and 9F are in Figures 5.9a, b, c, d, e and f respectively.

Table 5.9a

Option	G_k	Plausibility Trace
9A	[16, 18)	0.200
9B	[18, 20)	0.600
9C	[20, 22)	0.200
9D	[22, 23)	0.400
9E	[23, 24)	0.200
9F	[24, 26)	0.400

Figure 5.9a

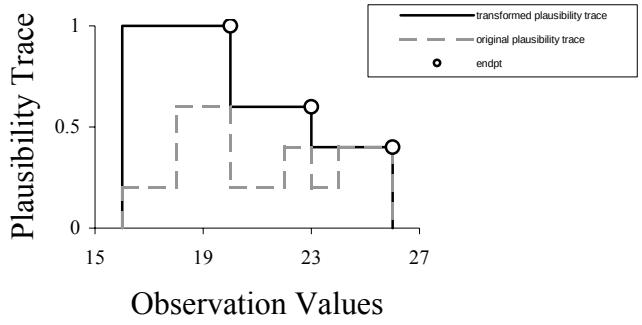


Figure 5.9b

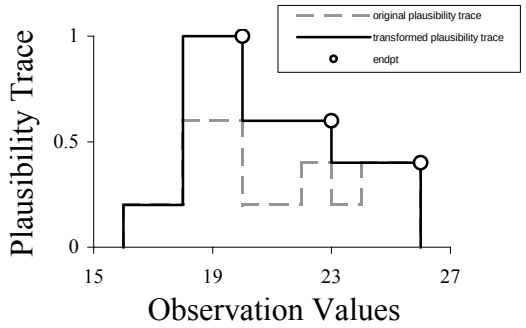


Figure 5.9c

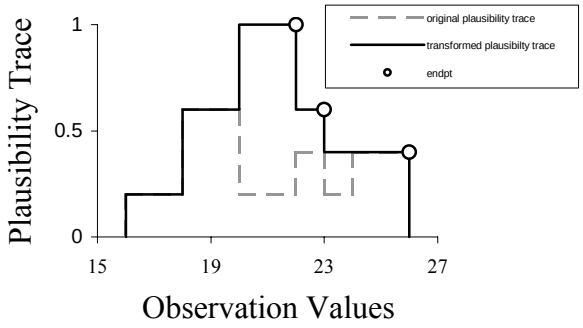


Figure 5.9d

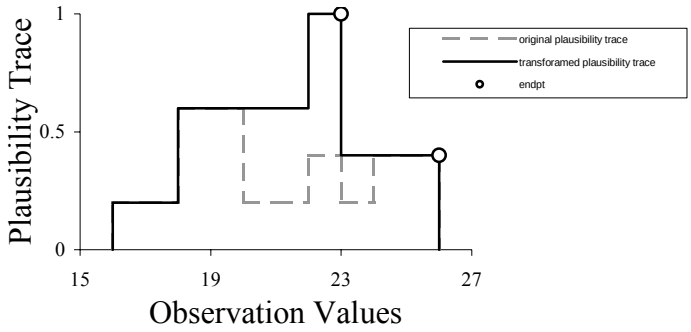


Figure 5.9e

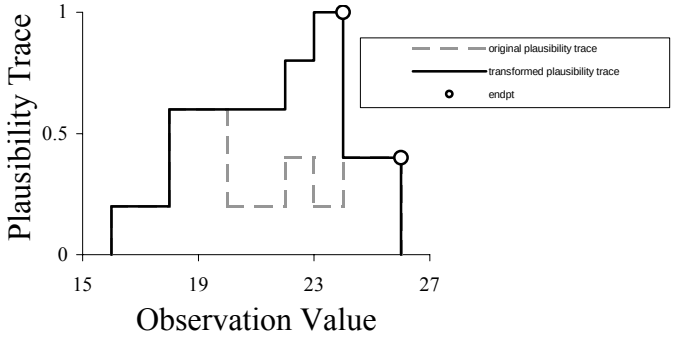


Figure 5.9f

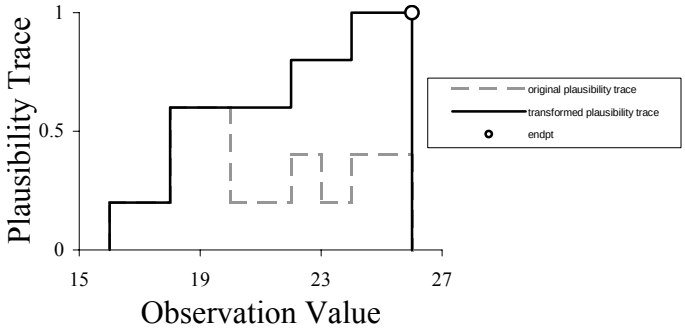


Table 5.9b

	N	U	ST	AU	Similarity	Distance
Option 9A*	2.690	2.690	1.123	3.322	0.600	3.600
Option 9B	2.129	2.161	1.420	3.171	0.700	2.000
Option 9C	2.395	2.156	1.632	3.322	0.700	2.000
Option 9D	2.290	1.729	2.138	3.322	0.700	**1.600**
Option 9E	2.351	1.982	1.473	3.322	0.700	2.000
Option 9F*	2.351	2.351	1.010	3.322	0.700	3.000

Again, based on minimal Hamming distance, the normalizing interval with the smallest size and maximum plausibility results in the transformed consistent set that is most like that of the original set (Option 9D).

Example 5.10

Consistent transformation of Example VI of Section 5.2 results in 4 alternative sets, these are shown in Table 5.10b. It is important to note that all the resulting transformed interval sets are consistent as well as consonant. These are formed from the normalizing intervals shown in Table 5.10a. The possibility distributions for these 4 alternatives are shown in Figures 5.10a, 5.10b, 5.10c, and 5.10d.

Table 5.10a

Option	G_k	Plausibility Trace
10A	[98, 99)	1/3
10B	[99, 102)	2/3
10C	[102, 106)	2/3
10D	[106, 107)	1/3

Figure 5.10a

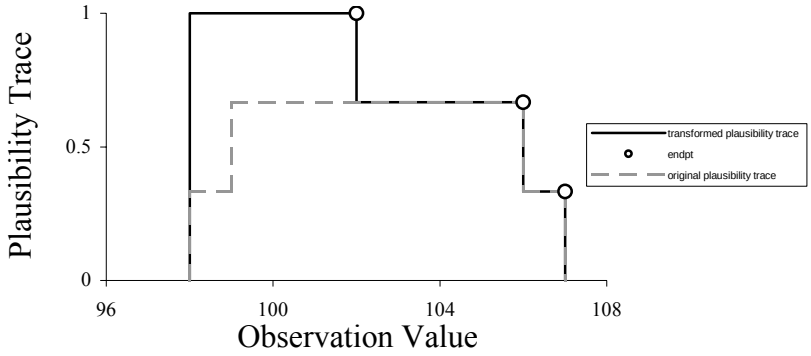


Figure 5.10b

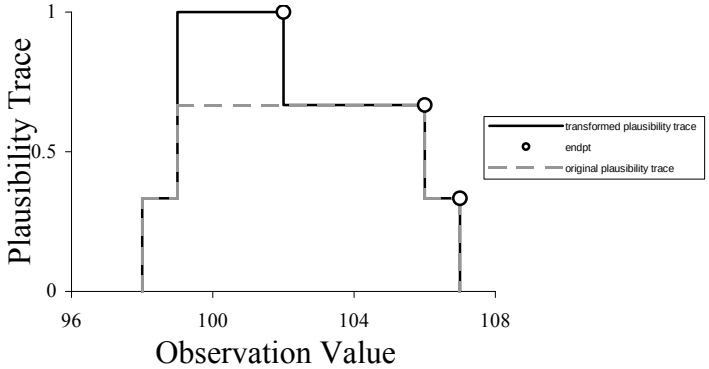


Figure 5.10c

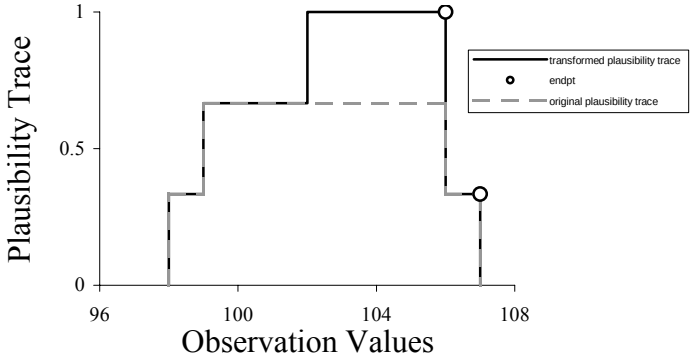


Figure 5.10d

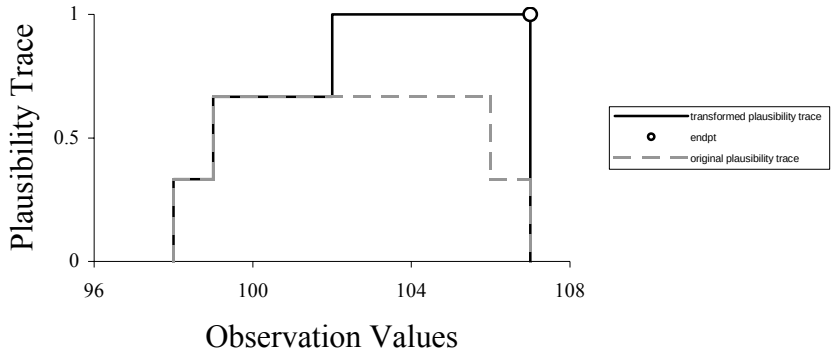


Table 5.10b

	N	U	ST	AU	Similarity	Distance
Option 10A*	2.723	2.723	0.918	3.170	5.333	1.667
Option 10B*	2.521	2.521	1.029	3.170	**5.333**	**1**
Option 10C*	2.659	2.659	0.938	3.170	5.333	1.333
Option 10D*	2.831	2.831	0.841	3.170	5.333	1.667

Based on Hamming distance, Option 10B is the transformed set that is most like the original set. Option 10B uses the smallest interval with maximal plausibility to normalize the original set.

5.5 Chapter Summary

This Chapter provides a variety of originally inconsistent, inconsistent and disjoint, and consistent examples and their transformation using the appropriate transformation algorithm (either Ross & Donald or Joslyn). This chapter also demonstrates the array of possible transformed sets that result for each untransformed set. Variability exists among the alternatives of this array of transformed sets, this is apparent not only in the resulting

possibility distribution but also in the uncertainty measurements of the resulting transformed sets. The differences between the original set and the transformed set can also be seen by comparing the plausibility trace of the original and transformed set shown in Appendix A.

The point of the examples is to develop a correlation between the normalizing intervals and the transformed set. The results of this chapter indicate that the size of the normalizing interval(s) plays an important role in obtaining a transformed set that is most like that of the original set. This is especially demonstrated in consistent transformation using Joslyn's [8, 9] transformation algorithm. It is also demonstrated in the smallest sub-interval of the normalizing intervals used for consonant transformation with Ross & Donald [4] algorithm. The results included in this chapter are essential to the automation of the transformation algorithms. The uncertainty measurements of Strife, Nonspecificity, U-uncertainty, and Aggregate Uncertainty can be used to further study the transformation algorithms. Their purpose in this chapter was to determine if there was a correlation between minimal uncertainty and the transformed set that was most like that of the original set; however, no correlation was determined.

Chapter 6

Implications of the Normalizing Interval(s) Characteristics on the Transformed Sets

6.1 Chapter Introduction

Chapter 5 demonstrates the possible transformation alternatives for each untransformed set from a collection of random sets, as well as the variability among these alternatives. Recall, that this collection of transformed sets is a consequence of the various normalizing intervals available for consistent transformation and the various consonant interval options available for consonant transformation. The transformed sets differ from one another; this is apparent, not only between resulting possibility histograms but the associated between conflict measures and nonspecificity measures as well. Chapter 5 also shows that whenever one uses a distance metric or a similarity metric, that there is a difference between the plausibility trace of the original untransformed set and that of the transformed sets. In each of the transformation algorithms, the normalizing interval(s) is required to transform the originally non-consonant and or non-consistent set. Thus, the characteristics of the normalizing interval or selected consonant normalizing intervals are related to the differences between transformed sets and their closeness to the original set. This chapter develops the relationship between the normalizing interval and consonant intervals and the "likeness" of the transformed set to the original set. This is

6.2 Essential Properties of the Transformed Set

The ambition here is to obtain a transformed set that is most “relevant” to the original untransformed set. In other words, the transformed set that is most like the original set is desired. This can be accomplished by comparing the plausibility trace of the original set to that of the transformed set. The plausibility trace of this transformed set that is most like that of the original set is the desired transformed set. At the same time, it is also desired that this transformed set will have a low uncertainty (Nonspecificity, Strife and Aggregate Uncertainty); however, this is treated secondary to the "likeness" between the two sets. As mentioned previously, there are various metrics available for determining the “likeness” between two distributions and, as discussed in Chapter 2, the Similarity metric as well as the Hamming distance metric are used here.

6.3 Consistent Transformation

In considering Examples 5.7 – 5.10 of Chapter 5, it appears that in almost every array of transformed sets the similarity measures are equivalent to one another. However, based only on the similarity measure it is impossible to determine which transformed set is “more like” the original untransformed set. Additional information is needed, which is provided by the Hamming distance metric. The transformed set with the minimum Hamming distance value is the "least different" from the original set. Hence, we conclude that the similarity metric chosen is not sufficient to determine which

transformed set is most like the original set. An additional metric, measuring likeness, is required to determine which transformed set is most like the original set. Therefore, the Hamming distance metric is used along with the Similarity metric.

For consistent transformation, Joslyn [8] has suggested selecting the normalizing interval based on maximal plausibility, which distorts the ordering of the original distribution as little as possible. Lets extend this further and base the selection on the smallest normalizing interval with the maximal plausibility. Considering the consistent transformation of the disjoint set of Example 5.7 in Chapter 5, Option 7B uses the smallest normalizing interval with the maximal plausibility trace. This results in a possibility distribution very similar to the original plausibility trace, as demonstrated with the Hamming distance measure which is a minimum for Option 7B. In Examples 5.8 and 5.10 of Chapter 5 a minimum Hamming distance value is again demonstrated. In both examples there exists two normalizing intervals with a maximal and identical plausibility trace however, the size of each is different. In both examples, the normalizing interval with minimal size results in a transformed set with a plausibility trace much like that of the original plausibility trace.

Now consider the overlapping set of Example 5.9, where Option 9B uses a normalizing interval with maximal plausibility. However, based on the plausibility trace, it does not produce the transformed set most like that of the original set. In this case, Option 9D uses a normalizing interval that is smaller in both plausibility and length than the normalizing interval of Option 9B, yet its transformed set is closer to that of the original.

Again, this seems to confirm that minimal "size" (minimal cardinality) is an essential selection criteria for the normalizing interval in consistent transformation. It appears that in consistent transformation, using the algorithm developed by Joslyn [8, 9], that size as well as the plausibility are related to the "likeness" of the transformed set to the original set. Hence, the normalizing interval should be selected based on maximal plausibility as well as minimal size.

6.4 Consonant Transformation

The selection of consonant intervals for consonant transformation is more complex than the selection of a single normalizing interval, as is required in consistent transformation. In this investigation we employ the "fast algorithm" developed by Ross & Donald [1] to assist in the elimination of inconsistencies and the selection of intervals to place in groups H and I. Levels are assigned to each interval based on their consonance with other intervals. Intervals that do not include any other intervals are assigned level 1 (or minimal level), the intervals that include level 1 intervals are assigned a level 2, intervals that include level 2 intervals are assigned level 3 and so on. With the use of these levels, consonant sub-collections are formed. As is the case for Example 5.1, we have two consonant sub-collections, each producing two different transformed sets. Again the task at hand is to determine the transformed set that is most like that of the original untransformed set. In this example, the two groups of selected consonant sets (Options) are identical with the exception of 1 interval in each consonant set ([20, 28) and [21, 29)).

Here Option 1A produces the plausibility distribution most like that of the original untransformed set. Interval [21, 29) has a greater plausibility than [20, 28). When conflicts such as this arise (where two intervals have equal levels and are not minimal) then it is first checked to determine which interval is included by more intervals; the interval that is included by more super-intervals is selected. If both are included by the same number super-intervals then the interval with maximal plausibility is selected. As is often the case, when equal plausibility exists, then the interval with the most common points is selected. This procedure produces a transformed set most like that of the original. This procedure is verified in the first 5 examples of Chapter 5.

Following the above procedure, for the overlapping set of Example 5.2, we are presented with a conflict between minimal intervals of Option 2B and Option 2D. For minimal intervals, the smaller interval with the max plausibility produces a transformed set most like that of the original set. This is demonstrated in Examples 5.2 and 5.6.

6.5 Nonspecificity and Strife

The Uncertainty is regarded as a secondary condition to the "likeness" between the transformed set. Two important components of Uncertainty are Strife (ST) and Nonspecificity (N); ST is a function for the measure of the conflict in evidence where as the function N measures nonspecificity in the evidence. Through consistent and consonant transformation the Strife of the set seems to be decreased which is expected. However, the Nonspecificity is often increased from that of the original set.

6.6 Chapter Summary

For consistent transformation, a normalizing interval with minimal length and maximal plausibility produces a transformed set that is closest to the original.

In consonant transformation, selection of the consonant set makes use of a fast algorithm proposed by Ross [1] to form the consonant set. Conflict among non-minimal consonant intervals is eliminated and is based on the interval's consonance in other sets as well as the plausibility and points in common with the rest of the set. Conflict among non-minimal intervals is based on minimum size and maximum plausibility. The uncertainty of the transformed sets is desired to be minimal, however it is secondary to the "likeness" requirement. Through the use of consistent or consonant transformation the Strife seems to decreased, and the resulting transformed sets have more agreement than the original set.

Chapter 7

Conclusions

7.1 Chapter Introduction

This chapter discusses the results of this investigation including essentials required to automate the transformation algorithms and eliminate analyst subjectivity. It goes on to discuss the contributions of this investigation on engineering applications where these algorithms are applicable. The chapter then points out that there should be no doubt with the findings of this investigation and their applications to continuous systems however this is an area for further study. An additional area for further study would be to compare the consonant transformed sets obtained from Jolsyn's algorithm [8, 9] with those obtained from the Donald [4] algorithm.

7.2 Automation of Transformation Algorithms

Chapter 6 explains the association between the characteristics of the normalizing interval and selected consonant intervals and the resulting consistent and consonant transformed sets, respectively. This association is determined using both the minimum Hamming distance as well as the maximum Similarity. In consistent transformation, the normalizing interval producing the transformed set that most resembles the information

portrayed by the original untransformed set is the normalizing interval with minimal length and maximal plausibility. In consonant transformation, the selected minimal consonant interval is selected in the same manner as the normalizing interval for consistent transformation. However, maximal consonant intervals are selected based on maximal consonance in other super-intervals, maximal plausibility and maximum points in common with the other intervals. Based on this criteria, the automation of the transformation algorithms is complete and analyst subjectivity is avoided.

The elimination of the subjectivity and automation of the algorithms allows them to be applied more efficiently and with less subjectivity. This is extremely important in engineering applications involving interval data, such as the buckling of hollow, metallic sphere presented in Chapter 1. The processing of possibility distributions for engineering applications consisting of a various types of interval data can now be accomplished in a less subjective and more efficient way.

7.3 Resulting Uncertainty

In regards to the uncertainty in evidence due to Strife, there is a decrease for consistently transformed sets and an even greater decrease for consonantly transformed sets. Generally, the Nonspecificity among evidence seems to increases for consonant transformation but occasionally there is a decrease. However, in consistent transformation there seems to be a general increase in Nonspecificity.

7.4 Discrete and Continuous Sets

Discrete interval sets are required for metrics and algorithms used in this investigation, however other metrics for determining the "likeness" between the original set and the transformed set could be used that do not require that the interval sets be discrete sets. In measuring the distance between two distributions of the same length formed from interval data of non-discrete interval sets, the distance is measured by modifying the Hamming distance metric where the summation becomes integration. The choice of using discrete interval sets was simply to conform to the metrics used in this study, the general utility of the findings are not limited to discrete interval sets.

7.5 Future Research

The examination of a larger variety of interval data would strengthen the results of this investigation, which should include continuous interval sets. Also, Jolsyn's transformation algorithm contains an additional step that forms a consonant set. In addition to examining a larger variety of interval data it would be beneficial to compare the resulting transformed consonant sets obtained using both algorithms. This could be accomplished by comparing the uncertainties and "likeness" for the resulting transformed sets from both algorithms. The findings of this research are valuable to the study of possibilistic normalizing methods, however solid conclusions are dependant on the results of an analytical study as well.

7.6 Long-Term use of Possibility Distributions in Reliability Assessments

Very recent studies [13] have shown that the possibility distribution method can ultimately be used as a guide in determining how well an analyst understands the extent of the relationship between modeling uncertainty and variability. A probability density function (pdf) reflects the amount of variability in a simulation. In contrast, the possibility distribution reflects the amount of *predictive uncertainty* (which is the sum of variability and modeling uncertainty) in the simulation. In Figure 1.3c one sees two different distributions that can be used to assess the differences between modeling uncertainty and variability. On the one hand, the predictive uncertainty and the variability could be almost the same if the aprons on the possibility distribution function have a near vertical slope (the pdf is a large part of the possibility distribution). On the other hand, the predictive uncertainty and the variability could be vastly disparate if the aprons on the possibility distribution function are have very low slopes (the pdf is a small part of the possibility distribution). Hence, in a sort of graphical way, the difference in the regions mapped by the variability (pdf) and the possibility distribution is a quantitative assessment of the modeling uncertainty in a problem. In this sense, the boundary regions of the possibility distribution can be used as a guide about where, specifically, we need more information in any planned future testing to reduce total uncertainty.

7.7 Chapter Summary

The results of this investigation describe the implications of the normalizing interval(s) characteristics on the transformed sets. The results of consonant transformation, using

the algorithm developed by Ross & Donald [4], suggest that the selection of minimal intervals should be based on maximal plausibility and minimal size. Whereas, the selection of non-minimal intervals should be based on the maximal inclusion of the interval in super-intervals, then maximum plausibility, and lastly number of discrete elements in common with other intervals of the focal set. The results of consistent transformation, using the algorithm developed by Joslyn [8, 9], suggest that the selection of the normalizing interval should be based on not only maximal plausibility but minimal size as well as. Automation of the transformation algorithms is accomplished by employing the results mentioned above. Automation removes the analyst subjectivity and produces a transformed set that is most like that of the original set.

The limitations made on the random intervals (discrete sets with elements defined at every integer) may have had an affect on the results. Further research should be conducted in this area to eliminate this doubt. The examination of a larger variety of interval data would also be beneficial in forming a complete investigation of this subject. The findings of this research are valuable to the study of possibilistic normalizing methods, however solid conclusions are dependant on the results of an analytical study as well.

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