M-Adaptation for Acoustic Wave Equation in 3D

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Numerical modeling of wave propagation is essential for a large number of applied problems in acoustics, elasticity, and electromagnetics. The acoustic equation is one of the simplest examples of equation modeling wave propagation. For long integration times, the dominant contributions to an error in the solution come from such numerical artifacts as numerical dispersion and numerical anisotropy.

Numerical dispersion is the phenomenon in which the propagation velocity of the wave in the numerical scheme depends on its wavelength, while in the continuum problem there is no such dependence. Typically, the effect of the *numerical dispersion* is greater on under-resolved waves with ten or fewer points per wavelength, making them travel slower than in the physical problem. As a consequence, the wave does not simply arrive at a wrong time (which could be compensated for by time rescaling), but also has a highly distorted profile. *Numerical anisotropy* is the dependence of the numerical velocity of the wave on its orientation with respect to the mesh. We developed an m-adaptation technique for the acoustic wave equation in 2D on rectangular meshes. We identified the optimal numerical schemes in a rich family of second-order Mimetic Finite Difference (MFD) schemes that are fourth-order accurate for the numerical dispersion. On square meshes these schemes could be further optimized to be sixth-order accurate for numerical anisotropy. We refined the m-adaptation technique to be used in 3D on cuboid meshes. We identified fourth-order accurate schemes in the family of second-order MFD schemes. The resulting schemes have a nearly optimal time-step stability condition within the family.

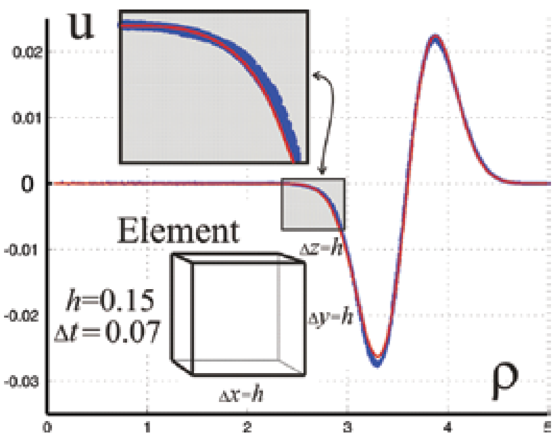

Fig. 1. Relative error in the numerical speed of the wave ch as a function of resolution parameter κh (number of points per wavelength $N=\frac{2\pi}{\kappa h}$) for various directions κ of the wave exp $(\vec{\kappa}\bullet\vec{x} - ct/\kappa)$. The m-adaptation on cuboid mesh (right) has the same fourth-order convergence rate as the modified quadrature method on cubic mesh (left).

The original and the semi-discrete forms of the acoustic wave equation in the time domain formulation are

$$u_{tt} = c\,\Delta u \text{ and } Mu_{tt} = Au \qquad (1)$$

where the mass and stiffness matrices M and A are assembled from elemental matrices M_E and A_E, and where c is the wave-speed. Since the mass matrix M has to be inverted on every time step, the explicit time discretization of equation (1) is computationally efficient only when the inverse M^{-1} is easy to compute. One of the approaches is to replace the mass matrix M with a diagonal matrix D by lumping nondiagonal entries to the diagonal. This does not change the order of the numerical scheme, but may lead to an undesirable increase of numerical dispersion.

Another approach [1] is to replace the inverse M^{-1} with the product $D^{-1}MD^{-1}$, where the inverse is taken only for the diagonal matrix D. Similar to lumping, this approach does not change the order of the numerical scheme. One can modify the stiffness and the mass matrices A and M using modified quadrature rules as is done in [2]. On square and cubic meshes, this approach produces schemes that are fourth-order accurate for numerical dispersion; however, this approach fails to do so in the more challenging case of rectangular and cuboid meshes.

Our approach has some similarities with [2] but is significantly more general. In fact, the schemes produced by [2] are a subset of the schemes analyzed in our approach, dubbed m-adaptation. We consider a parameterized MFD family of numerical schemes from which we select a member with the smallest numerical dispersion and anisotropy. The parameters in the MFD family appear through the elemental mass and stiffness matrices M_E^{MFD} and A_E^{MFD}, respectively. In 3D, the number of parameters is significantly larger than in 2D, so the techniques used in 2D are no longer tractable. For example, in 2D on a rectangular mesh the elemental matrices M_E^{MFD} and A_E^{MFD} depend on two parameters and one parameter ζ, respectively. In 3D on cuboid meshes the elemental matrices M_E^{MFD} and A_E^{MFD} depend on 28 and 10 parameters, respectively.

The MFD family parameterized by $(m_i \zeta_j)$ contains a large number of known methods as special cases, for example, standard Finite Difference (FD), rotated FD, weighted combination of standard and rotated FD, Finite Element (FE) with lumped mass matrix, and modified quadrature method [2]. Moreover, compared with the later method, the MFD family is richer–containing 36 more parameters.

For the acoustic wave equation in 2D we selected the optimal parameters $(m_1 m_2, \zeta)$ based on the von-Neumann analysis. In 3D, due to a much larger number of parameters, this approach was no longer tractable. We replaced this approach with another one, where we cancel the errors

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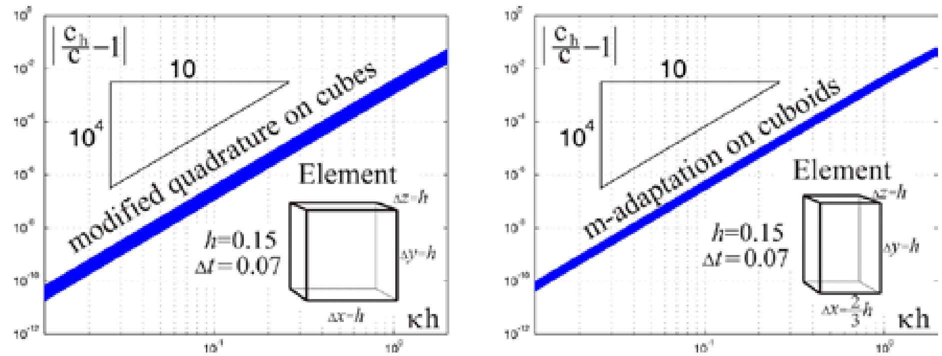
coming from the spatial and temporal discretizations for plane polynomial waves $(\vec{\kappa}\vec{x}\omega\tau)^p$ of degrees $p = 1,\ldots,4$ for all possible directions and magnitudes of vector $\vec{\kappa}$. The seemingly infinite set of conditions for canceling the two errors can be condensed into a system of 21 equations that depend bilinearly on the elemental mass and stiffness matrices (thus on the parameters). The current state of the art for solutions for systems of bilinear equations does not allow for writing the solution for the system in an explicit form. Therefore, we have to rely on numerical solution of the system.

In addition to satisfying the above-mentioned 21 conditions, the set of 38 parameters has to yield positive definite mass and stiffness matrices. Moreover, the largest time step for which the scheme is numerically stable is inversely proportional to the largest eigenvalues of the matrices. Thus, we have to control both the largest and smallest eigenvalues of the mass and stiffness matrices. We identified and implemented an iterative numerical procedure for the solution of the above system subject to the optimization of eigenvalues. Based on this procedure, we found numerical schemes among second-order accurate MFD schemes that are fourth-order accurate for numerical dispersion both on cubic and non-cubic cuboid meshes.

We tested the optimized schemes using the dispersion relation, which in 3D has a similar form to the one we obtained in 2D for rectangular meshes. Once presented in logarithmic scale for the relative numerical error, it clearly shows fourth-order accuracy for the numerical dispersion (Fig. 1). This is the same accuracy as obtained by the modified quadrature scheme [2], but now it is achieved on general cuboid meshes. Moreover, on cubic meshes–although we obtained the same fourth-order accurate schemes as the modified quadrature schemes–our schemes had a stable time step that was larger by at least 10%.

As another test, we simulated a radially symmetric wave spreading from the origin, starting with Gaussian displacement and zero initial velocity. The radial symmetry



tests the numerical anisotropy, while the Gaussian profile (containing all wave frequencies) tests the numerical dispersion. The test shows that the optimized scheme on cuboid mesh $\Delta x = .1$, $\Delta y = \Delta z = .15$, with $\Delta t = .07$ has comparable dispersion to that of the modified quadrature scheme on a cubic mesh $\Delta x = \Delta y = \Delta z = .15$ with $\Delta t = .07$ and both produce very little dispersion for a mean wavelength of the Gaussian corresponding to 12 points per wavelength ($\kappa h \approx 0.5$).

Fig. 2. Displacement as a function of the distance from the origin at time $T = 3.6$ obtained using the modified quadrature method on cubic mesh (left) and the m-adaptation method on cuboid mesh (right) for a Gaussian initial displacement data.

In the future, we plan to develop the m-adaptation technique for higher order schemes on general meshes and for elastic wave equations. The advantages of m-adaptation are that at a cost of some preprocessing one finds a fourth-order accurate scheme that has the complexity of a second-order one, requiring no matrix inversion during time step, therefore making it very efficient and accurate at the same time.

[1] Gyrya, V. and K. Lipnikov, "M-Adaptation Method for Acoustic Wave Equation on Square Meshes," LA-UR 12-10047; *J Comput Acoust* **20(4),** 1250022:1-23 (2012).

[2] Guddati, M.N. and B. Yue, *Comput Meth Appl Mech Eng* **193,** 275 (2004).