Parallel Algorithm for Spherical Centroidal Voronoi Tessellations

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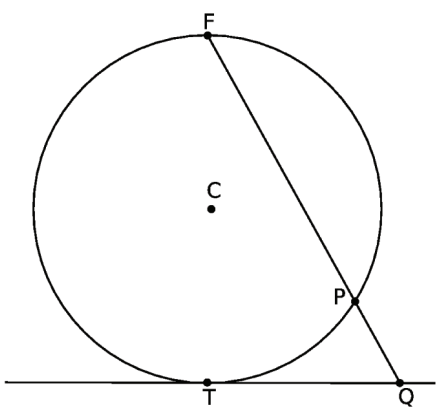
Spherical centroidal Voronoi tessellations (SCVT) are used in many applications in a variety of fields, one being climate modeling. They are a natural choice for spatial discretizations on the Earth, or any spherical surface. The climate modeling community, which has started to make use of SCVTs, is beginning to focus on exascale computing for large-scale climate simulations. As the data size increases, the efficiency of the grid generator becomes extremely important. Current high resolution simulations on the earth call for a spatial resolution of about 15 km. In terms of an SCVT this corresponds to a quasi-uniform SCVT with roughly 2 million Voronoi cells. Computing this grid serially is very expensive and can take on the order of weeks to converge sufficiently for the needs of climate modelers. Outlined here is a new algorithm that utilizes existing computational geometry tools such as conformal mapping techniques, planar triangulation algorithms, and basic domain decomposition, to compute SCVTs in parallel, thus reducing the overall time to convergence. This new algorithm shows speedup on the order of 4000 when using 42 processors over STRIPACK in computing a triangulation used for generating an SCVT.

Recent developments in ocean and atmospheric modeling allow for simulation on Spherical Centroidal Voronoi Tessellations (SCVT) [1]. However, SCVTs cannot simply be prescribed, they have to be generated. In order to generate a SCVT, algorithms make use of their dual mesh, Delaunay Triangulations. Typically SCVTs are generated using what is known as Lloyd's algorithm, which can be either deterministic of probabilistic [2]. A simple version of this algorithm would be as follows:

1) Start with a point set
2) Determine the Delaunay Triangulation of the point set
3) Determine the Voronoi Diagram of the point set
4) Determine the center of mass for each Voronoi Region
5) Replace the point set with the centers of mass
6) Iterate until converged

A Voronoi diagram defines regions where all points contained inside of a region are closer to the region's center than they are to any other point in the generating set. A special type of Voronoi diagram is known as a centroidal Voronoi tessellation, which occurs when the center of mass of a Voronoi region is coincident with the region's generating point. The spherical variant of a centroidal Voronoi tessellation is the same as a planar version, but on the surface of a sphere. Current software for determining a spherical Voronoi tessellation, such as STRIPACK, are limited in the size of meshes they can compute as well as the speed at which they can determine the tessellation.

Fig. 1. Cross-section of a stereo-graphic projection from a sphere into a plane.



In order to generate high resolution meshes for ocean and atmospheric simulations, a new algorithm was developed to compute SCVTs in parallel. In order to generate a SCVT in parallel two things are done with the point set. First, domain decomposition needs to be used. In order to simplify this process, a SCVT that is coarser than the target SCVT is used to decompose the surface of the sphere into regions. After this is done, tangent planes are defined at each of the region centers. Using these tangent planes, points within a region are stereo-graphically projected (Fig. 1) into the plane and the Delaunay triangulation is computed in a plane. This triangulation can then be mapped onto the surface of the sphere with only small modifications, including the removal of triangles whose circumcircles extend outside of the region's radius. This triangulation is valid because stereographic projections preserve circularity, keeping the insides and outsides of circles through the projection. The only restrictions in order for a triangle to be Delaunay involve its circumcircle being empty, and having three or more points on its perimeter.

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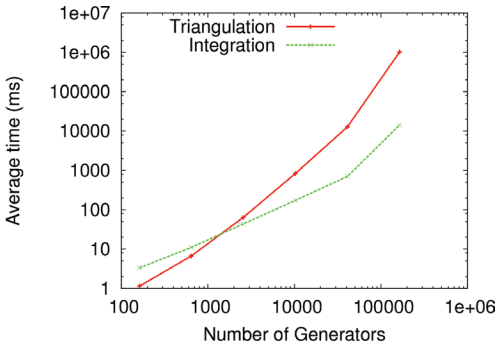


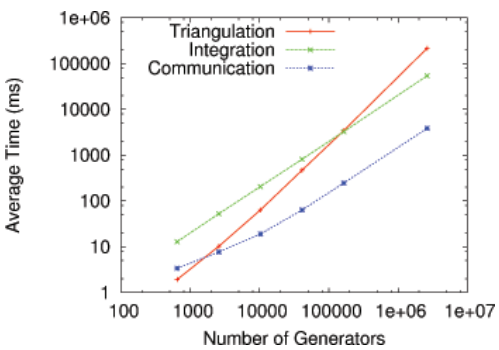
Fig. 2. Timing for different portions of a STRIPACK-based SCVT generator.

Using these techniques the new algorithm for generating an SCVT in parallel is as follows:

1) Start with a point set
2) Decompose point set into regions
3) Stereographically project point sets into tangent planes
4) Triangulate points in tangent planes
5) Map triangulations onto sphere
6) Determine centers of mass of Voronoi regions
7) Replace point set with centers of mass
8) Iterated until converged

During the decompose step, a point might exist in more than one region. This causes each region to overlap with its neighbors and adds a small buffer region that ensures the interior of the region contains the exact Delaunay triangulation and Voronoi diagram for the points that need to be updated. A naive approach is to define a radius for each region, and sort each point using a dot product with the center. If the dot product results in a distance that is smaller than the region radius that point will be part of that region's computational domain. This overlap is the reason that certain triangles need to be removed from the set. If a triangle's circumcircle extends outside of its region radius, points outside of the current set might be contained inside of the circumcircle, making it no longer Delaunay. After points are triangulated, a region only updates points contained within its Voronoi region.

Fig. 3. Timing for different portions of a modified SCVT generator, using two processors.



This modified version of Lloyd's algorithm can now be used to generate SCVTs in parallel using distributed memory systems. An implementation of this algorithm is compared with an implementation of Lloyd's algorithm using STRIPACK for serial triangulations. When using the serial version, the cost of computing a triangulation increases non-linearly as the problem size increases (Fig. 2). Whereas the parallel version provides linear increases in both the cost of triangulation and the cost of computing the centers of mass when using only two processors (Fig. 3). Adding more processors increases the speedup when compared with the STRIPACK version to something on the order of O(4000), and this algorithm shows reasonable scaling with when generating a 2.6-million-point mesh (Fig. 4).

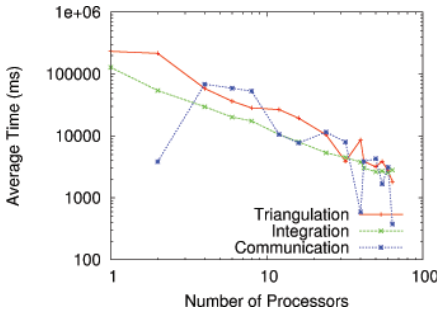


Fig. 4. Scaling for the modified SCVT generator using 2621442 generators.

One issue that arises when exploring the generation of variable resolution meshes with this algorithm amounts to poor load balancing. This problem is fixed by first sorting points into Voronoi regions for the coarse SCVT. After this initial sorting is complete, a regions point set consists of the union of the points contained in that region and all of that neighboring regions.

Using this new algorithm, high-resolution meshes can now be generated in parallel on the surface of the sphere for use in ocean and atmosphere models. In addition to providing the capabilities of high resolution mesh generation, lower-resolution meshes can be generated in significantly less time than previous techniques provided. This method can also be ported to be used in a plane by removing the stereographic projection portion, and it may be possible to also extend it to allow parallel triangulations of an entire 3D sphere using some other techniques.

[1] Ringler, T.D. et al., *Mon Weather Rev* **139,** 3348 (2011).
[2] Lloyd, S. *IEEE Trans Inform Theor* **28(2)**, 129 (1982).

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