Discrete Vector and Tensor Calculus on Polyhedral Meshes

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A DVTC builds discrete operators that preserve fundamental identities of continuum calculus. As a result, important mathematical and physical principles such as conservation laws and solution symmetries are easily incorporated in numerical models. In our recent paper [1], we completed development of a DVTC on arbitrary polyhedral meshes. The developed DVTC ensures that discrete models are consistent approximations of continuum models, leading to more accurate predictions.

Polyhedral meshes provide enormous flexibility for describing complex engineering models that arise in simulating dispersive transport in porous media, scattering of elastic waves from irregular interfaces in seismic imaging, and representing realistic surface topography in climate modeling. The developed discrete vector and tensor calculus (DVTC) ensures that discrete models are consistent approximations of continuum models, leading to more accurate predictions.

Polyhedral meshes provide a path to a robust mesh generation and adaptation. These meshes eliminate the critical problems of tetrahedral meshes. Polyhedra are less sensitive to stretching than tetrahedra; therefore, they are more suitable for problems with boundary layers. Slivers, typical for major tetrahedral mesh generators, are easily eliminated by merging them with two or more tetrahedra. Adaptive and non-matching meshes, popular in engineering applications, result only in different types of polyhedra; therefore, no special treatment of hanging nodes is required for discretizations and solvers.

Polyhedra have many more neighbors than tetrahedra–for example, Kelvin's tetrakaidehedron has 14 faces and is considered the ideal prototype of a bubble in a dry, monodisperse foam. From one side, this increases the stencils of discrete operators, but from the other side, this makes the operators more accurate, reduces numerical diffusion, and allows information to propagate faster through the mesh, leading to an increased overall convergence rate. Finally, polyhedra cover the space more efficiently than tetrahedra and hexahedra and tend to minimize the inter-element surface area.

The DVTC is at the core of all mimetic discretization methods [1-3]. The numerical solution of a heat conduction problem is frequently performed using degrees of freedom (DOF) at mesh nodes. With these DOF, we may approximate the temperature gradient along each edge via a conventional finite difference. This results in a primary mimetic gradient operator, $grad_h$, that acts from the space of node-based discrete functions to the space of edge-based discrete functions. The DVTC builds a dual of this operator, the discrete divergence operator, $\widetilde{div}_h$, that acts in the opposite direction [2]. The nodal-based discretization of the Laplace operator is $\widetilde{div}_h grad_h$.

To enforce local mass balance in a diffusion problem, its mixed formulation is often used. Discretization of the divergence operator becomes trivial when face-based unknowns approximate normal components of flux. This results in a primary mimetic divergence operator, div_h, that acts from the space of face-based discrete functions to the space of element-based discrete functions. The DVTC builds a dual discrete gradient operator, $\widetilde{grad}_h$, that acts in the opposite direction [3]. The cell-centered discretization of the Laplace operator is $div_h \widetilde{grad}_h$.

For the Maxwell equations, natural DOF for the electric field are its tangential components on mesh edges. Such a definition of DOF results in a simple primary mimetic curl operator, $curl_h$, that acts from the space of edge-based discrete functions to the space of face-based discrete functions. The DVTC builds a dual of this operator, $\widetilde{curl}_h$.

The DVTC preserves important mathematical identities such as $\widetilde{curl}_h \widetilde{grad}_h = 0$ and $div_h \, curl_h = 0$ and provides discrete Helmholtz

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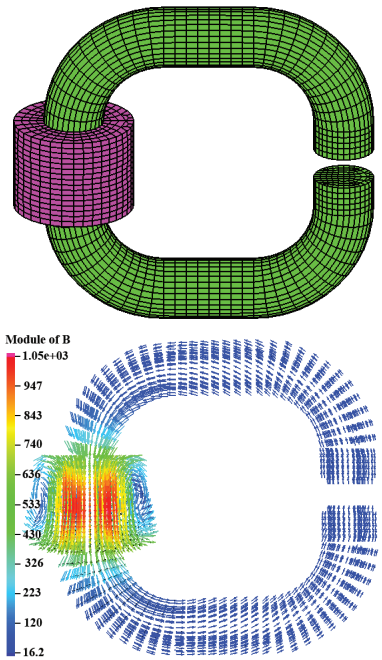

Fig. 1. A model of the C-shaped magnet consists of a copper slab wrapped around the core of a ferromagnetic material. The core is a cylinder of electric steel bent to form a C-shape. The core enhances the magnetic field produced by the circular current running in the copper. The model is meshed with a quasi-uniform hexahedral mesh (using package CUBIT) with about 50,000 elements. The top picture shows a trace of the computational mesh. The bottom picture shows the magnetic induction. The arrows plotted at mesh nodes indicate the expected alignment of the magnetic field with the ferromagnetic core. Solution of this problem with equivalent tetrahedral mesh would require twice the unknowns.

decomposition theorems. Note that dual operators are not unique, which explains the existence of various approaches to the development of a DVTC. Unique features of our approach are (1) the absence of constraints on the shape of polyhedra, and (2) consistency of arguments embedded in all derivations to guarantee accuracy of dual operators on arbitrary polyhedral meshes.

Future work will be focused on the development of an arbitrary-order accurate DVTC.

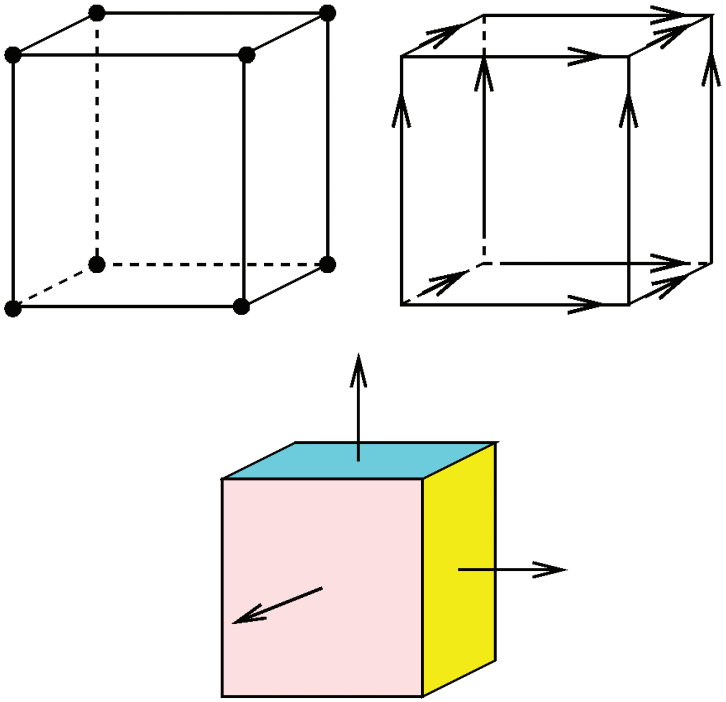

Fig. 2. Location of node-based (top left), edge-based (top right), and face-based (bottom) degrees of freedom for a hexahedron.

[1] Lipnikov, K., et al., *J Comput Phys* **230,** 305 (2011).
[2] Brezzi, F., et al., *M2AN, Math Model Numer Ana* **43,** 277 (2009).
[3] Brezzi, F., et al., *Comput Meth Appl Mech Eng* **196,** 3682 (2007).

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