High-order Divergence-free MHD Solver for Turbulence Simulation

Shengtai Li, T-5

Turbulence is a common phenomenon in plasmas. It is convenient to treat turbulent plasmas in a magneto-hydrodynamic (MHD) framework. MHD studies the dynamics of electrically conducting fluids. Magnetic fields have a nice property called divergence-free condition, which means that the divergence of the magnetic fields is always zero, for conservation of the magnetic flux.

The MHD turbulence simulations demand methods with both high-order accuracy and a divergence-free preserving property. The order of accuracy, which is also called the rate of convergence of simulation results to the true solutions, is one of main factors of the efficiency of the method. The low-order method demands a much finer grid resolution than the high-order method in order to achieve the same accuracy. Numerical experiments show that the eighth-order scheme can save CPU time by a factor of 85, and data storage by a factor of 64, over the second-order scheme in achieving a comparable effect in turbulence simulations. The divergence-free preserving method is also essential in order for MHD simulations to produce the correct dynamics (see Fig. 1). Due to the lack of highly accurate simulation software, the research and development on compressible MHD turbulence is still in its infancy. We are aware of no higher than a second-order divergence-free method thus far in MHD simulations.

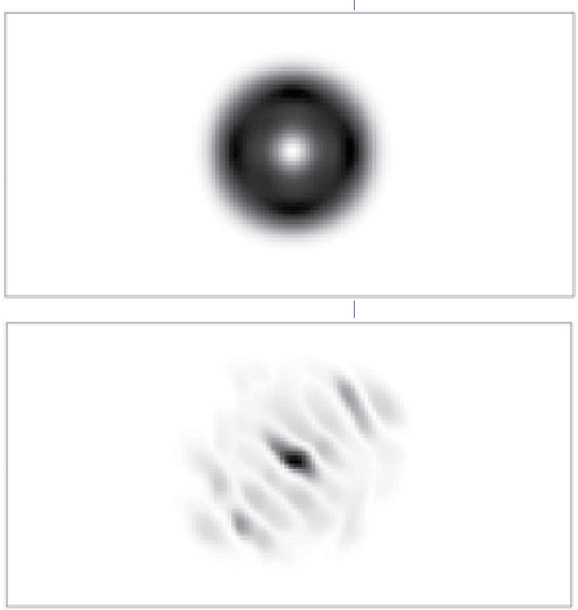

Fig. 1. The divergence-free method (top) preserves the magnetic field-loop better than the nondivergence-free method (bottom).

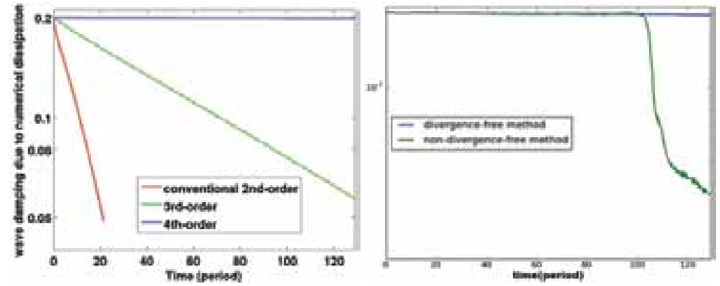

Fig. 2. The left plot shows the fourth-order method has much lower numerical dissipation than the third- or traditional second-order methods. The right plot shows the nondivergence-free method can introduce large dissipation at certain times during simlations.

The traditional numerical simulations use only one grid to cover the whole physical domain. The magnetic field at the cell boundaries can numerically become discontinuous due to the numerical interpolation. The unphysical discontinuity of the magnetic field can trigger numerical instability [1] and crash the simulation.

Inspired by the work in [2], we propose a new methodology to develop numerical methods for MHD simulations. Rather than using only one grid, we generate an overlapping dual grid from the primal grid. By solving the MHD equations on both the primal and dual grids, the cell boundaries of one grid fall naturally within the dual grid, the magnetic fields become continuous in the dual grid, and the flux is easy to calculate by using the dual grid information. The accuracy order of the method is equal to the order of the reconstruction on a cell, which can be arbitrarily-high order accuracy. The divergence-free condition can also be easily achieved by developing high-order, divergence-free reconstruction over a cell.

We first developed a third-order divergence-free method for ideal MHD in 2D [3]. With the help of a dual grid, we get rid of the cell-centered magnetic field and related spatial averaging, which is needed by conventional method. We can compute the magnetic field with arbitrarily high-order accuracy. We have constructed a compact third-order, divergence-free scheme for MHD simulations. To our knowledge, this is the first verified higher than second-order scheme

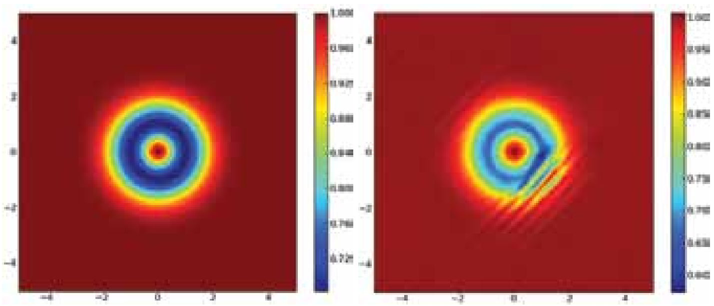

Fig. 3. The divergence-free method preserves the vortex very well, whereas the nondivergence-free method introduces a large error in the vortex.

with the divergence-free property for MHD simulations.
For MHD flows that contain shock and contact discontinuity, we proposed an essentially nonoscillatory reconstruction to preserve the monotonicity of the fluid variables. This limited reconstruction uses only adjacent nearest neighbors via a hierarchical procedure.

We have also developed a 4th-order method using the same framework as the 3rd-order method [4]. The 4th-order, divergence-free reconstruction requires a constraint in addition to the dual overlapping grid. Numerical verification shows that our method achieves 4th-order accuracy and is divergence-free for the magnetic fields. We have found that the 4th-order method is roughly 32 times more efficient than the 3rd-order method in order to achieve the same accuracy as for 2D problems. Figure 2 shows the 4th-order method has much lower numerical dissipation than the 3rd-order method and conventional 2nd-order method.

We have found that the divergence-free property is crucial in our high-order method, especially when limited reconstruction is used. The right plot of Fig. 2 shows that without the divergence-free condition, the wave is suddenly damped due to the large divergence error in the numerical method. Figure 3 shows the pressure for a vortex advection in a periodic domain. After five periods, the vortex solved by the divergence-free method is preserved very well, whereas the nondivergence-free method has a large error produced.

Figure 4 shows that three linear waves propagate and interact with each other before they become turbulence. The nondivergence-free solution has a lot of numerical noise in the wave profile, which is introduced by a large divergence error in the magnetic field.

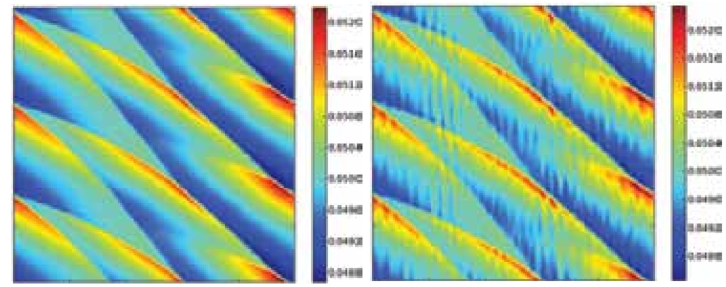

Fig. 4. The nondivergence-free method (right) introduces much noise to the solutions in the linear waves interaction and propagation problem, whereas the divergence-free method (left) retains a clean wave profile.

We have extended our 2D 3rd- and 4th-order methods to 3D. The extension is not trivial and will be documented in our future paper.
Numerical verifications show that we have achieved the expected order of accuracy for both the divergence-free and nondivergence-free methods.

Comparing the efficiency and accuracy of the 3rd- and 4th-order methods for 3D, we have found that the 4th-order method is 64 times more efficient than the 3rd-order method in order to achieve the same accuracy for 3D problems.

For more information contact Shengtai Li at sli@lanl.gov.

[1] T.J. Barth, *IMA volume on Compatible Discretization,* Springer-Verlag (2005).
[2] Y. Liu et al., *Commun. Comput. Phys.* **2,** 933 (2007).
[3] S. Li, *JCP* **227,** 7368 (2008).
[4] S. Li, "The fourth-order divergence-free method for magneto-hydrodynamic flows," (to be submitted to *JCP*)

Funding Acknowledgments
LANL Directed Research and Development Program