Grid Generation and Adaptation Using an L_p form of Monge-Kantorovoch Optimization

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Grid generation and adaptation by Monge-Kantorovich optimization [1] produces optimal grids that equidistribute the local grid error, thereby minimizing the total error [2]. Optimality [1] was defined as minimizing the L_2 norm

$$\|\mathbf{x}'-\mathbf{x}\|_2 = \left[\tfrac{1}{2}\int |\mathbf{x}'-\mathbf{x}|^2\rho(\mathbf{x})d\mathbf{x}\right]^{1/2}$$

with the local constraint that the map $x' = x'(x)$ has Jacobian $\det[\partial(x')/\partial(x)]$ equal to $\rho(x)/\rho'(x')$, where $\rho(x)$ and $\rho'(x')$ are the local errors at successive time steps (initial and target densities). In [1] it was shown that this method approximately minimizes the distortion of the map (the trace of the covariant metric tensor), and that the resulting equation, the Monge-Ampère equation, can be effectively solved by multigrid preconditioned Newton-Krylov methods.

In [3] we extended this approach to more general domains in 2D and to 3D, showing that the advantages detailed in [1] apply to these more general cases.

We have recently investigated the L_p form of this error equidistribution problem. That is, we minimize

$$\|\mathbf{x}'-\mathbf{x}\|_p = \left[\frac{1}{p}\int |\mathbf{x}'-\mathbf{x}|^p\rho(\mathbf{x})d\mathbf{x}\right]^{1/p}$$

with the same Jacobian (equidistribution) constraint $\det[\partial(x')/\partial(x)] = \rho(x)/\rho'(x')$, for $p > 1$. The motivation for considering $p \neq 2$ is that, relative to $||x' - x||_p$ for $1 < p < 2$, $||x' - x||_2$ penalizes very little for small excursions but greatly for large excursions. Also, there has been a considerable amount of work in the mathematics literature for $p = 1$, involving the use of the deformation method, discussed in [1]. Minimizing $||x'-x||_p$ with the Jacobian constraint, we conclude that

$$\mathbf{x}' = \mathbf{x} + |\nabla\mu|^{\frac{2-p}{p-1}}\nabla\mu(\mathbf{x}) \qquad (1)$$

where $\mu(x)$ is the Legendre transform of a Lagrange multiplier enforcing the Jacobian constraint. The form corresponding to (1) for $p = 2$ is $x' = x + \nabla\mu(x)$. In the difficult limit $p \to 1$, we find that $\mu(x)$ satisfies the Eikonal equation $|\nabla\mu| = 1$. Substituting equation (1) into the Jacobian condition, we obtain a generalized Monge-Ampère equation for $\mu(x)$. We have found that this equation can also be solved by Newton-Krylov methods.

In Fig. 1 we show a target density $\rho'(x')$, which is peaked at $x = 0.7$, $y = 0.9$, and the ratio of the maximum to the minimum density is ~ 22. The initial density is $\rho(x) = 1$. In Fig. 2 we show the grid x' obtained with these densities for a uniform initial grid x, using various values for p. For $p \gtrsim 1.5$, the grids seem reasonable, concentrated near the peak of ρ' and with little distortion, i.e., the cells are close to square.

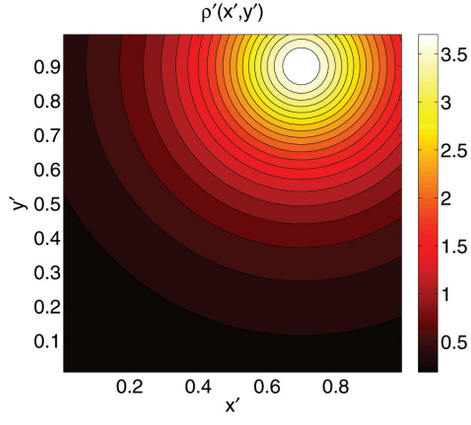

Fig. 1. Target density.

For $p = 2$, the grid lines are orthogonal to the boundary. For $p < 1.5$, and especially for $p \to 1$, some of the cells are seriously distorted and stretch from the origin to the peak of ρ' at $x = 0.7$, $y = 0.9$. (The darker grid lines have $x = 0.25$,

$\downarrow p_c \backslash p_m \rightarrow$	1	1.01	1.02	1.05	1.1	1.25	1.5	1.75	2	2.25	2.5	Distortion
1.01	0.0000	0	0.0000	0.0004	0.0018	0.0103	0.0320	0.0572	0.0833	0.1093	0.1348	0.2501
1.02	0.0001	0.0000	0	0.0002	0.0013	0.0091	0.0295	0.0534	0.0782	0.1029	0.1270	0.2367
1.05	0.0006	0.0004	0.0002	0	0.0005	0.0063	0.0236	0.0444	0.0661	0.0878	0.1089	0.2130
1.1	0.0021	0.0017	0.0013	0.0005	0	0.0031	0.0162	0.0330	0.0510	0.0690	0.0867	0.1687
1.25	0.0092	0.0084	0.0076	0.0054	0.0028	0	0.0043	0.0132	0.0238	0.0351	0.0464	0.0716
1.5	0.0223	0.0210	0.0198	0.0164	0.0118	0.0036	0	0.0020	0.0063	0.0118	0.0178	0.0158
1.75	0.0334	0.0319	0.0304	0.0264	0.0208	0.0096	0.0017	0	0.0011	0.0035	0.0068	0.0018
2	0.0425	0.0408	0.0392	0.0348	0.0285	0.0155	0.0050	0.0010	0	0.0006	0.0022	0.0014
2.25	0.0499	0.0482	0.0464	0.0417	0.0349	0.0207	0.0085	0.0029	0.0006	0	0.0004	0.0069
2.5	0.0561	0.0542	0.0525	0.0475	0.0404	0.0254	0.0118	0.0052	0.0019	0.0004	0	0.0154

Table 1. Values of the p_m norm and the mean distortion for calculations with $p = p_c$

x = 0.75, y = 0.25, y = 0.75 and are shown to emphasize the distortion for small p.) Further, the grid lines are far from orthogonal to the boundary. This adverse behavior near the boundary and especially near the corner (0, 0) for $p \rightarrow 1$ is traced to the fact that for $p = 1$, μ satisfies the Eikonal equation, which leads to a boundary layer for p close to unity.

More quantitative measures of the grid quality are contained in Table 1. The rows are labeled by p_c, which specifies the value of p used in the computations. The columns are labeled by p_m, the value of p for which the norm is measured. For example, for the highlighted in red entry with $p_c = 1.01$ and $p_m = 2$, we computed the map solving the generalized Monge-Ampère equation with $p = p_c$, but measured $||x'-x||_p$ for $p = p_m$. (The norms are scaled to the diagonal value with $p_m = p_c$ with unity subtracted.) Note that in all cases, the terms below the diagonal are smaller than the transposed values $p_m \leftrightarrow p_c$. That is, computing the grid with large p_c works well even when quality is measured by small p_m, whereas computing with small p_c leads to large norms $||x'-x||_{p_m}$ for large p_m. It appears that $p_c = 2$ works well, but from this measure larger p_c seems slightly better. Note, however, that the measured distortion (mean of the trace of the metric tensor) is minimal at $p_c = 2$, orders of magnitude larger for small p_c and a factor of 10 larger for $p = 2.5$. Overall, these results indicate that $p_c = 2$ leads to the highest quality grids.

Based on these results, we conclude that grid generation and adaptation works best with $p = 2$. We have also observed that the domain of attraction of Newton's method is largest for $p = 2$ and becomes much smaller for p close to unity. Further, there is a well-developed preconditioning strategy for $p = 2$ [1].

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[1] G.L. Delzanno et al., *J. Comput. Phys.* **227**, 9841 (2008).
[2] G. Lapenta, *J. Comput. Phys.* **193,** 159 (2004).
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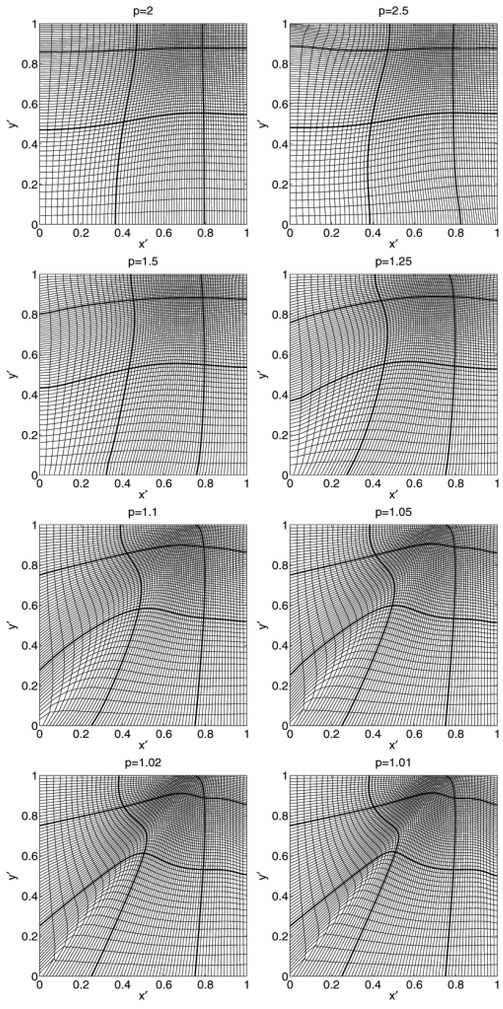

Fig. 2. Grids for various p.

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