Fourier Analysis of Parallel Block-Jacobi Splitting with Transport Synthetic Acceleration in Two-Dimensional Geometry

Massimiliano Rosa, James S. Warsa, Jae H. Chang, CCS-2

Fourier analysis is traditionally used to study transport iteration schemes in a homogeneous infinite medium. In fact, it represents a valuable tool to understand the behavior of the iteration error modes of various acceleration techniques, either in their continuous or spatially discretized forms. A Fourier analysis has been recently performed in slab geometry [1] for the discrete-ordinates (S_N) approximation of the steady-state one-group transport problem solved with Richardson iteration and preconditioned Richardson iteration, using the Parallel Block-Jacobi (PBJ) algorithm. Two types of Transport Synthetic Acceleration (TSA) have been considered as a preconditioner, the traditional "Beta" TSA (TTSA) and a modified TSA (MTSA).

The Fourier analysis for PBJ and for PBJ with TTSA and MTSA is extended to 2-D Cartesian geometry. The spatial discretization considered is a Bilinear Discontinuous Finite Element Method (BLDFEM) and the scattering is assumed to be isotropic. The analysis is verified with results from a 2-D transport code. As stated in [1], the results of the Fourier analysis reveal the effectiveness of iteration algorithms in terms of spectral radius ρ and minimum eigenvalue Λ of the symmetric part of the iteration matrix. The latter are functions of the quadrature order (number of angles), total macroscopic cross section σ and scattering ratio *c*.

First we look at the results of the Fourier analysis for PBJ without any preconditioning using S_4 level-symmetric quadrature, $\sigma = 1$ and $c = 0.5$. Results show that convergence of PBJ algorithm can degrade for problems containing optically thin subdomains, even for values of the scattering ratio *c* less than unity. In fact the spectral radius, as the cell widths are decreased, is independent from the value of *c*. Furthermore, the minimum eigenvalue Λ of the symmetric part of the iteration matrix may become negative indicating that restarted GMRES, GMRES(m), may stagnate for optically thin problems.

The predictions from the Fourier analysis for PBJ have been compared with the numerical results obtained from a 2-D transport code. To reproduce the conditions of the Fourier analysis, the 2-D transport code ran on four processors with four computational cells per processor (Nx = Ny = 2) and with reflective boundary conditions on all four boundaries, in order to eliminate the effect of leakage. The results for ρ and Λ obtained for a level-symmetric S_4 quadrature with equal weights, a unitary total macroscopic cross-section, and a scattering ratio of 0.5 are compared with the Fourier analysis. In both cases the numerical results are very close to the predicted theoretical values.

Next we examine the number of iterations and the residual for numerical calculation using GMRES(m) with different values of the restart parameter. The maximum number of iterations is set to 1000 and the error tolerance is 10^{-5}. The results show that GMRES(m) may stagnate for a value of the restart parameter equal to the square root of the number of processors, which is number of steps needed to propagate the information across the full problem domain. This finding is in agreement with negative minimum eigenvalues obtained in the Fourier analysis.

The results above indicate that preconditioning of the PBJ algorithm is needed to improve its spectral properties, especially for optically thin problems. Therefore we conducted a Fourier analysis of PBJ accelerated using both traditional TSA (TTSA) and modified TSA (MTSA) as preconditioners in 2-D Cartesian geometry. Since the effective scattering ratio *c'* is a measure of the actual computational effort required by each acceleration method, the comparison of the two methods is carried out for equal values of this parameter. In particular, the results presented refer to the case $c' = 0$ ($\beta = 1$), which is an upper bound for the spectral radius.

The surface plots are obtained by plotting the spectral radius ρ from the Fourier analysis as a function of the cell widths, Fig. 1 for TTSA and Fig. 2 for MTSA, respectively. TTSA appears extremely effective for thick problems, for which the spectral radius tends to zero. However, the spectral properties degrade for thin problems. Also, for values of the scattering ratio *c* greater than 0.5, the spectral radius may become greater than unity leading to instability, unless the β parameter is properly selected. MTSA is slightly less effective than TTSA for thick problems, where the spectral radius is slightly greater than zero. The spectral properties are better for thin problems. As a matter of fact, the spectral radius is always bounded from above by the value of the scattering ratio *c*, for $\beta = 1$. MTSA is therefore unconditionally stable.

A Fourier analysis has been implemented for the Parallel Block-Jacobi (PBJ) algorithm and for PBJ with both traditional TSA (TTSA) and modified TSA (MTSA). The results for the unaccelerated algorithm show that convergence of PBJ can degrade and lead to stagnation of GMRES(m) in problems with optically thin subdomains. These predictions have in turn been successfully verified against a 2-D transport code.

Fourier analysis for the accelerated PBJ algorithm indicates that PBJ with TTSA can be effective provided the β parameter is properly tuned for a given scattering ratio *c*, but is potentially unstable. Compared to TTSA, MTSA is less sensitive to the choice of β, more effective for the same computational effort (c'), and it is unconditionally stable.

For more information contact Jae H. Chang at jhchang@lanl.gov.

[1] M. Rosa, et al., "Fourier Analysis of Parallel Inexact Block-Jacobi Splitting with Transport Synthetic Acceleration in Slab Geometry," in *Proceedings of the American Nuclear Society's Topical Meeting on Reactor Physics: PHYSOR-2006,* Vancouver, BC, Canada, September 10–14, 2006 (American Nuclear Society, 2006).

Funding Acknowledgements
NNSA's Advanced Simulation and Computing (ASC), Computational Physics and Methods Strategic Capability, Advanced Transport Discretizations, and Iterative Methods Project.

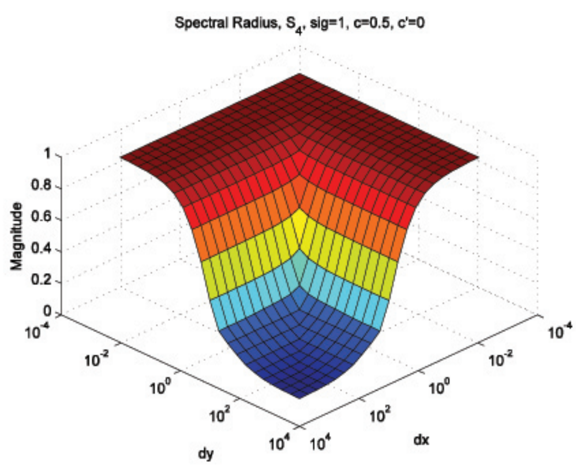

Fig. 1.
Fourier analysis for PBJ with TTSA: ρ.

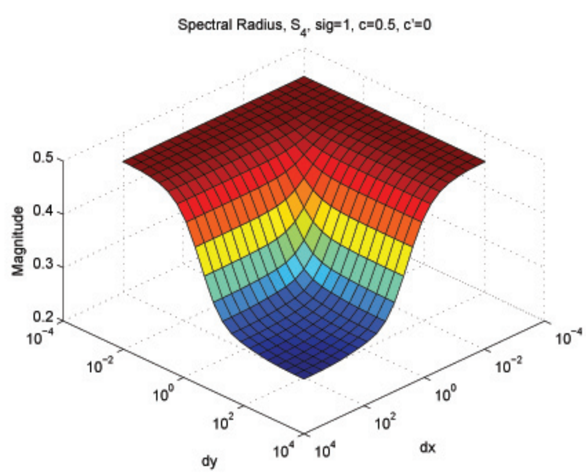

Fig. 2.
Fourier analysis for PBJ with MTSA: ρ.

Los Alamos NATIONAL LABORATORY EST. 1943