

Analysis of the Stability, Monotonicity, and Time Discretization Error of the Implicit Monte Carlo Method

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The Implicit Monte Carlo (IMC) method for nonlinear, time-dependent, radiative transfer calculations employs a semi-implicit approximation to the photon absorption-emission process within a time step [1]. This approximation originates from a backward-Euler time discretization of the exact equations followed by a linearization of the radiation-material coupling. An IMC calculation proceeds in a series of time steps, each of which consist of a linear radiation transport calculation followed by a nonlinear material energy update. Although the semi-implicit time discretization allows the use of larger time steps than a purely explicit technique, the method can produce nonphysical solutions if the time step is too large [2]. We are motivated to understand the limitations of this approximation in order to ultimately develop an *a priori* time-step control algorithm that avoids nonphysical behavior.

To this end we have studied gray (frequency-independent), 0-D (space-independent) problems, in which the radiation and material energy densities initially coexist in a nonequilibrium state. As energy is exchanged between the material medium and radiation field (via photon absorption and emission) the system approaches thermal equilibrium asymptotically in time. The time-dependent radiation and material energy densities are described by a differential radiation transport equation that is coupled to a differential material energy density equation. The equations can be either linearly or nonlinearly coupled, depending on the form of the material heat capacity. For the linear case, we have developed conditions that guarantee that the exact IMC solution (neglecting statistical error) is stable (i.e., progresses toward the analytic equilibrium solution) and monotone

(i.e., does so without nonphysical oscillations) [3]. For the nonlinear case, we have obtained an exact solution and have used it to numerically estimate the time order of accuracy of the IMC method [4].

For the linear case, the IMC method generates end of time-step energy densities through a linear transformation of the beginning of time step values

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n \quad (1)$$

where \mathbf{x} denotes the radiation and material energies. If the time step is constant, then \mathbf{A} is a constant 2×2 matrix of coefficients for the problem at hand. The time stability and monotonicity of the IMC approximation can be investigated by expanding the solution of the IMC equations in the eigenvectors of the \mathbf{A} matrix

$$\mathbf{x}_n = a\lambda_1\varphi_1 + b\lambda_2^n\varphi_2. \quad (2)$$

where λ and φ denote the eigenvalues and eigenvectors of \mathbf{A} , respectively, and a and b are constants. The first term on the right-hand side is the exact equilibrium state of the system. The second term represents the transient behavior between the initial condition and thermal equilibrium. Since $\lambda_1 = 1$, the method is stable whenever $|\lambda_2| < 1$ and monotonic whenever $0 \leq \lambda_2 < 1$. From these conditions, time-step limits have been derived that guarantee stability and monotonicity of the method. These limits have been verified numerically. In Fig. 1, the IMC energy densities are plotted (in arbitrary energy and time units) for a problem in which the time step was chosen to be half of the stability limit. It is clear that the solution is stable but not monotonic and hence nonphysical. In Fig. 2, the time step was chosen to be half of the monotonicity limit. The exact solution is also plotted, and it can be seen that the IMC method yields a physically valid result.

For the nonlinear case, stability and monotonicity limits are much more difficult to obtain since the \mathbf{A} matrix is no longer constant in time. However, a solution to the exact radiative transfer equations has been obtained (whereas most existing benchmarks have been developed for linear problems). The solution has been used to benchmark the IMC approximation and determine its order of accuracy with respect to the time variable. For that purpose, we examine the integrated error in the IMC-predicted time-dependent material temperature

$$\text{RMS Error} = \sqrt{\frac{1}{N} \sum_{n=1}^N [T_{n+1} - T(t_{n+1})]^2}, \quad (3)$$

where T_{n+1} and $T(t_{n+1})$ denote the IMC and exact end of time-step material temperatures, respectively. In Fig. 3, the RMS error is plotted vs time-step size for $\alpha = 0, 0.5,$ and 1 . Here, α (where $0 \leq \alpha \leq 1$) is a user-selected parameter that controls the time-centering of the absorption-reemission approximation. As expected, the numerical results indicate that the method is first-order accurate in time-step size for $\alpha = 0$ and 1 , and second-order accurate for $\alpha = 0.5$. It should be noted that IMC absorption-reemission approximation is most stable when $\alpha = 1$, and often this is the best choice.

Through the analysis of idealized radiative transfer problems, we have learned much about the stability, monotonicity, and accuracy of the IMC method. In future work, we intend to use these results as a guide to develop *a priori* time step controls for more complex radiative transfer problems.

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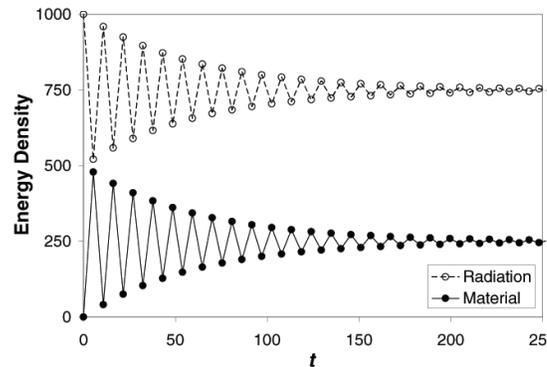


Fig. 1. IMC energy densities for a time step at half of the stability limit.

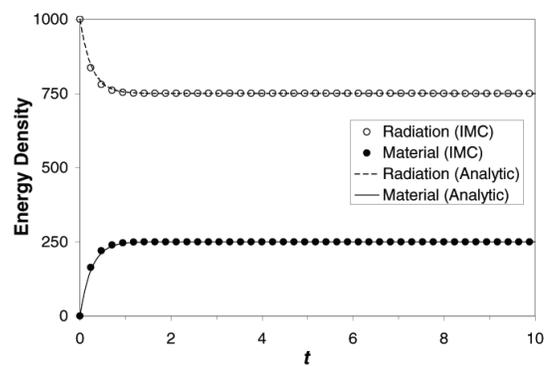


Fig. 2. IMC energy densities for a time step at half of the monotonicity limit.

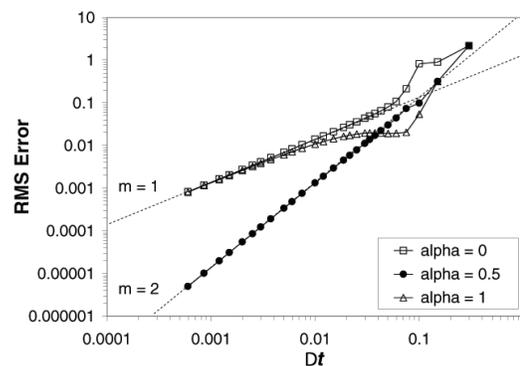


Fig. 3. IMC material temperature RMS error.