An acoustic streaming instability in thermoacoustic devices utilizing jet pumps
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Thermoacoustic-Stirling hybrid engines and feedback pulse tube refrigerators can utilize jet pumps to suppress streaming that would otherwise cause large heat leaks and reduced efficiency. It is desirable to use jet pumps to suppress streaming because they do not introduce moving parts such as bellows or membranes. In most cases, this form of streaming suppression works reliably. However, in some cases, the streaming suppression has been found to be unstable. Using a simple model of the acoustics in the regenerators and jet pumps of these devices, a stability criterion is derived that predicts when jet pumps can reliably suppress streaming. © 2003 Acoustical Society of America. [DOI: 10.1121/1.1543588]

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I. INTRODUCTION

Recently, thermoacoustic-Stirling hybrid engines 1–4 and refrigerators 4,5 that utilize traveling-wave phasing have been investigated. These devices are composed of a sandwich of three heat exchangers embedded in a looped acoustic network. The heat exchangers include an ambient heat exchanger, a stack or regenerator, and a heat exchanger at the "working" temperature which is above or below ambient depending on whether the device is an engine or refrigerator. These devices are filled with a thermodynamic working fluid, typically a pressurized ideal gas. The looped acoustic network may have distributed impedances, i.e., may have propagation lengths on the order of an acoustic wavelength, 1 or it may have lumped elements, i.e., with lengths much shorter than 1/4 of an acoustic wavelength. 1,2,4,5 In this article, the focus will be on engines and refrigerators that utilize regenerators embedded in a lumped-element acoustic network, because an engine of this type has already demonstrated high efficiency and refrigerators of this type promise higher efficiencies than existing orifice pulse tube refrigerators (OPTRs) at noncryogenic temperatures. A schematic drawing of a general lumped-element device is shown in Fig. 1. These devices are referred to as thermoacoustic-Stirling hybrid engines (TASHEs) and feedback pulse tube refrigerators (FPTRs).

Both TASHEs and FPTRs rely on their lumped acoustic networks for two roles. First, the network allows acoustic power to feed back from the working-temperature end of the regenerator to the ambient end. In an OPTR, this power is dissipated in an acoustic resistance at room temperature, while the acoustic network of an FPTR recycles this power into the ambient end of the regenerator. 5 This reduces the input power requirement and increases the efficiency of an FPTR compared to an OPTR. 6 In a TASHE, the acoustic power circulating in the acoustic network takes the place of mechanical components in various types of Stirling engines, such as cranks, linkages, or pistons, reducing the mechanical complexity of the engine. 2 In both devices, the second role of the network is to set the magnitude and phase of the acoustic impedance in the regenerator. By shifting the phase of the velocity oscillation by ~90°, the network transforms the standing-wave phasing in the resonator into nearly traveling-wave phasing in the regenerator. Also, the network keeps the magnitude of the acoustic impedance high so that viscous losses in the regenerator are reduced to an acceptable level. 2,5

Although the lumped acoustic network makes both the TASHE and FPTR possible, it also introduces a complication. The network is essentially a wide-open tube that connects one end of the regenerator to the other. This topology allows for acoustic streaming, a steady flow of mass, around the looped network and through the regenerator. In fact, the acoustic power circulating around the looped network inherently encourages such streaming in the direction from the ambient heat exchanger, through the regenerator, and to the working heat exchanger. A steady mass flow, no matter its direction, causes a heat leak to (FPTR) or from (TASHE) the working heat exchanger, drastically lowering the efficiency of both the TASHE and FPTR.

To combat this, a membrane or bellows could be used to block streaming around the loop while permitting oscillatory flow, but this might compromise the reliability inherent in a device that would otherwise have no moving parts. Alternatively, a nonlinear device termed a jet pump can be used to generate a time-averaged pressure, imposing low pressure at the ambient end of the regenerator so as to oppose the inherent tendency to stream. Typically, the geometry of the jet pump can be adjusted until the absence of streaming is indicated by a nearly linear temperature distribution through the regenerator. 2,5 In all TASHEs 2,8–10 that we have constructed to date, this technique has worked reliably. In contrast, one FPTR 11 that we have constructed demonstrated peculiar behavior. In that FPTR, it was impossible to adjust the geometry of the jet pump to stably cancel the streaming mass flux. By observing the temperature distribution in the regenerator, it was clear that there was always vigorous acoustic stream-

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The instability is investigated using a typical linear perturbation approach, where the acoustic variables are taken to be their equilibrium values plus a small perturbation. Using the usual acoustic expansion and time-harmonic notation, the steady-state solution can be written

\[ p(x,t) = p_m + \text{Re} \left[ p_1(x) e^{i\omega t} \right] + p_{2,0}(x), \]

\[ U(x,t) = \text{Re} \left[ U_1(x) e^{i\omega t} \right] + U_{2,0}(x), \]

\[ T(x,t) = T_m(x) + \text{Re} \left[ T_1(x) e^{i\omega t} \right], \]

where \( p, U, \) and \( T \) are the gas pressure, volumetric flow rate, and temperature, respectively, variables with the subscript 1 are complex, and \( \text{Re} \) signifies the real part. The subscript 2.0 indicates a second-order time-independent quantity. The angular frequency of the oscillation is \( \omega, \) \( t \) is time, and \( x \) is the coordinate along the axis of the regenerator, with \( x = 0 \) at the ambient face and \( x = x_w \) at the working-temperature face. If the perimeter of the regenerator is well insulated, the steady-state second-order energy flux, \( H_2, \) is independent of both \( x \) and \( t. \)

A simplified model of the acoustics in the regenerator, which we summarize here, has been used to obtain relationships between the various steady-state terms. If the compliance, or void volume, of the regenerator is neglected, and the temperature dependences of density and the temperature dependence of gas density require that the volumetric flow rate \( U_1 \) be given by

\[ U_1(x) = U_1(0) \frac{T_m(x)}{T_a}, \]

where \( T_a = T_{m}(0) \) is the mean temperature of the gas at the ambient end of the regenerator. Throughout the rest of this article, the subscripts “a” and “w” refer to a mean variable evaluated at the ambient and working-temperature faces of the regenerator, respectively. The volumetric flow rate into the ambient end, \( U_1(0), \) is estimated using a simplified model of the feedback network, yielding

\[ U_1(0) = \frac{\omega^2 LC}{R_m} p_1(0), \]

where \( L \) and \( C \) are the inertance and compliance in the feedback loop, respectively. Typically, \( \omega L R_m^{-1} \) and terms of order \((\omega L R_m)^{-2}\) have been dropped to simplify the calculation. Here, \( R_m \) is the flow resistance of the regenerator referenced to \( U_1(0), \) i.e.,

\[ R_m = \frac{\Delta p_{1,\text{regen}}}{U_1(0)} = \frac{R_0}{x_w} \int_{x_w}^{x_a} \left( \frac{T_m(x)}{T_a} \right)^{1+b} dx, \]

where \( R_0 \) is the flow resistance of the regenerator when its entire length \( x_w \) is at \( T_a \) and \( \Delta p_{1,\text{regen}} \) is the change in \( p_1 \) across the regenerator. The precise value of \( R_0 \) is not important in this calculation. The “1+b” in the exponent \((1+b)\) takes into account the \( T_m \) dependence of density and \( U_1 \) expressed in Eq. (4), and the “b” in the exponent takes into account the temperature-dependent viscosity of the gas in the regenerator, i.e., \( \mu(T_m) = \mu(T_a)[T_m(x)/T_a]^{b}. \) The expression for \( R_m \) is valid in the low-Reynolds-number limit for screen beds, parallel plates, and other typical regenerator geometries. In most cases, \( |\Delta p_{1,\text{regen}}| \ll |p_1|, \) and, therefore, \( p_1(x) \approx p_1. \) Without loss of generality, \( p_1 \) can be taken to be real. Equations (4) and (5) show that \( U_1(x) \) is also real. Other symbols with subscript 1 will continue to represent complex variables.
The steady-state time-averaged second-order mass flux \( M_{2,0} \) is given by
\[
M_{2,0} = \text{Re} [\rho_1 U_1] / 2 + \rho_m U_{2,0},
\]
where \( \rho_m \) is the mean density of gas in the regenerator, \( \rho_1 \) is the first-order complex density oscillation amplitude, and the tilde denotes complex conjugation. Inside the regenerator, the pressure oscillations are nearly isothermal. Neglecting the first-order complex density oscillation amplitude, and the second term in \( M_{2,0} \) can be written \( (\rho_m / \rho_m^2) \rho_1 U_1 / 2 \), so that \( M_{2,0} = (\rho_m / \rho_m^2) \rho_1 U_1 / 2 + \rho_m U_{2,0} \).

The second term in \( M_{2,0} \) depends on the second-order, steady volumetric flow rate \( U_{2,0} \), which is driven by gradients in the second-order steady pressure \( p_{2,0} \) and by source terms that involve time averages of two first-order quantities. Waxler has shown that the latter source terms are negligible compared to the gradients in \( p_{2,0} \) in a parallel-plate regenerator, and we assume that this holds true for other regenerator-pore geometries. The second-order momentum equation is then identical to the first-order momentum equation with \( d/dt, \) \( U_{2,0}, \) and \( p_{2,0} \) replacing \( i \omega, U_1, \) and \( \rho_1, \) respectively.

In a TASHE or FPTR, there can be several sources of \( p_{2,0} \) including, but not limited to, pipe bends, diffusers, and etc. However, in a well-designed device, the major source of \( p_{2,0} \) will be the jet pump used to control the streaming. In the steady state, \( M_{2,0} = 0 \) and \( p_1(x) \) is assumed to be equal to \( p_1(t) \), requiring \( U_{2,0} \) to have the same spatial and \( T_m(x) \) dependence as \( U_1 \). Therefore, the same definition of \( R_m \) also applies to \( U_{2,0} \), i.e., \( R_m = \Delta p_{2,0} / U_{2,0} \). Using these two results, \( M_{2,0} \) at the ambient end of the regenerator can be written
\[
\dot{M}_{2,0}(0) = \frac{\rho_m}{2 \rho_m} \rho_1 U_1(0) + \rho_m \frac{\Delta p_{2,0}}{R_m}.
\]

Since \( \partial M_{2,0} / \partial x = 0 \), this also gives the value of \( M_{2,0} \) throughout the regenerator. Although \( M_{2,0} = 0 \), Eq. (7) is required later when the steady-state solution is perturbed. If the jet pump is located near the ambient end of the regenerator, the time-average pressure drop across the regenerator, \( \Delta p_{2,0} \), will be given by
\[
\Delta p_{2,0} = \frac{\rho_m (U_1(0))^2}{8 A_{jp}} (K_{\text{out}} - K_{\text{in}}),
\]
where \( A_{jp} \) is the area of the small opening in the jet pump, and \( K_{\text{in}} \) and \( K_{\text{out}} \) are the minor loss coefficients for the two directions of flow through the jet pump. If the jet pump were located near the working-temperature heat exchanger instead of the ambient end, an additional multiplicative factor of \( T_m / T_a \) would appear in Eq. (9). As will be seen later [see, e.g., Eq. (36) and the surrounding discussion], any multiplicative scale factor in \( \Delta p_{2,0} \) that is unaffected by the perturbation does not affect the final result of the calculation.

III. PERTURBED SOLUTION

Next, an exponentially growing or decaying perturbation is added to the equilibrium solution reviewed in the previous section, so that the complete solution is of the form
\[
p(x,t) = p_m + p_1 \cos \omega t + p_{2,0}(x) + \delta p_{2,0} \cos \omega t.
\]

The perturbation includes both oscillating and nonoscillating terms. The oscillating terms are assumed to have the same frequency as the corresponding terms in the equilibrium solution and an amplitude that changes slowly, but exponentially, in time compared to the acoustic period, i.e., \( |\epsilon| < \omega \).

This two-time-scale approach allows the explicit separation of the slow change of the instability from the rapid acoustic oscillations. Nonoscillating terms also change exponentially in time with the same time constant as the amplitudes of the oscillating terms. Note that \( \delta \) can be taken to be real for the same reasons that \( U_1 \) is taken to be real.

We assume that \( T_a \) and \( T_m \) are fixed by good heat exchange with external fluids that have high heat capacity or latent heat or with high-heat-capacity heat exchangers. In either case, the thermal reservoirs in or near the heat exchangers are large enough to adjust to a varying heat load without a significant change in temperature. We also assume \( \delta p_1 \) and \( \delta \omega \) are zero. For an FPTR, these last two conditions can be regarded as simple consequences of how the system is driven, e.g., by a linear motor at fixed frequency and whatever electric current is required to keep \( p_1 \) fixed. For a TASHE, one can similarly assume that the complex load impedance is deliberately varied to keep \( \omega \) and \( p_1 \) fixed.

The calculation begins with the energy flux \( \dot{H} \) through the regenerator. In the steady state of a well-insulated regenerator, \( dH/dx = 0 \), meaning there is no build up of energy inside the regenerator. If a small, time-dependent perturbation is present, this condition is relaxed and the energy equation takes the form
\[
\frac{\partial}{\partial x} \left( \frac{1}{1 - \phi} \rho_s c_s T_s A + \rho_m c_v T_m A \right) = - \frac{\partial \dot{H}}{\partial x},
\]

where \( \rho_s, c_s, \phi, A, \) and \( T_s \) are the regenerator solid density, heat capacity, porosity, cross-sectional area, and temperature, respectively, and \( c_v \) is the isochoric heat capacity of the gas. Essentially, this equation states that if there is a spatial variation in the energy flux through the regenerator, energy is accumulated or depleted temporally, resulting in a temperature change of the working gas and regenerator solid at that location. On the acoustic time scale, the temperature of the solid and gas are not necessarily equal. (In fact, oscillations of the gas temperature give rise to the major source of \( \dot{H} \).) However, on the slow time scale of the perturbation growth, the excellent thermal contact between the regenerator solid and the gas will enforce \( \delta T_s \approx \delta T_m \). Also, in a typical regenerator, the heat capacity of the solid is much higher than the gas, i.e., \( \phi \rho_m c_v / (1 - \phi) \rho_s c_s \approx 1 \). Substituting the equilib-
rium solution plus perturbation into Eq. (15) and using the two approximations above yields, at linear order in the perturbation,

$$\epsilon(1-\phi)\rho_c e_A \delta T_m - \frac{d \delta H}{dx} = 0.$$  \hspace{1cm} (16)

A typical solution of Eq. (16) would proceed as follows. The steady-state solution plus perturbation would be substituted into the other governing equations, such as the momentum and continuity equations, and a system of differential equations relating $\delta H(x)$ and $\delta T_m(x)$ would be obtained. Next, a set of eigenfunctions would be found that satisfies both Eq. (16) and any boundary conditions imposed at $x = 0$ and $x = x_w$. These eigenfunctions would then be substituted back into Eq. (16) to determine the growth rate $\epsilon$ for each eigenfunction. Conditions under which one of the growth rates becomes positive would indicate an instability. Although this procedure would find all possible conditions for linear instability, carrying it out in this case would prove quite difficult. The resulting system of differential equations is not one with constant coefficients and is quite complicated. Finding a full set of eigenfunctions would be tedious, if not impossible, without numerical computation. Also, the difficulty of the calculation would obscure the physical effects that cause the instability.

To avoid this difficulty, a much simpler approach that yields analytical results is used, although it does not explore all possible instabilities. First, Eq. (16) is integrated from $x = 0$ to $x = x_w$, eliminating the need to know the detailed form of $d \delta H/dx$. Information about $\delta H$ is only required at the ends of the regenerator. Next, the $x$ dependence of the temperature perturbation is taken to be

$$\delta T_m = \sum_{n=1}^{\infty} \Theta_n \sin(n \pi x/x_w).$$ \hspace{1cm} (17)

Implicit in Eq. (17) is our assumption that the heat exchangers at either end of the regenerator hold the faces at $x = 0$ and $x = x_w$ at their steady-state values. It should be noted that the individual terms in Eq. (17) are not eigenfunctions of this problem, so there will be interaction between the various terms. However, this interaction will not be included in the following analysis, and only the $n = 1$ term will be carried through. (Both of these simplifications are reasonable in light of the observed temperature distribution in the FPTR that demonstrated a streaming instability: The deviation from a linear temperature distribution resembled half of a sine wave.\textsuperscript{11}) Substituting Eq. (17) into Eq. (16), setting $\Theta_1 = \Theta$, and integrating from $x = 0$ to $x = x_w$ yields

$$\frac{2x_w}{\pi} \rho_c e_A(1-\phi)A \epsilon \Theta - [\delta H(0) - \delta H(x_w)] = 0,$$ \hspace{1cm} (18)

where any temperature dependence of $\rho_c e_A$ has been ignored.

Typically, regenerators are made from piles of stainless-steel mesh. However, to estimate which terms of $\delta H$ are important, the analytic expression for $\delta H$ in a parallel-plate regenerator will be used. Except for axial conduction through the regenerator solid, the various terms are expected to have the same order-of-magnitude values in a screen-based regenerator. The steady-state energy flux consists of four terms\textsuperscript{14}

$$\frac{\delta H(0)}{\delta T_m(0)} = \frac{\text{Re}}{2} \text{Re} \left[ 1 - \frac{f_n - \bar{f}_n}{(1 + \sigma)(1 - \bar{f}_n)} \right]$$

$$+ \frac{\rho_n c_p U_l^2}{2 \sigma A} \text{Im} \left( f_n + \sigma \bar{f}_n \right) \frac{d T_m}{dx}$$

$$- \left[ \phi A k + (1 - \phi) A_k \right] \frac{d T_m}{dx} + M_{2\theta} c_p T_m$$

$$= H_b + H_c + H_{\theta} + H_m,$$ \hspace{1cm} (19)

where each term has been given its own symbol for convenience, and the fact that $U_l$ is real has been used to simplify the expression. In the assumed steady-state solution, $M_{2\theta} = 0$, and therefore $H_m = 0$. However, $\delta M_{2\theta}$ and its associated enthalpy flux, $\delta H_m$, need not be zero. Since $\delta T_m(0) = \delta T_m(x_w) = 0$, the gas properties and $f$ functions at $x = 0$ and $x = x_w$ do not change, and $\delta H(0)$ can be written

$$\delta H(0) = \frac{\delta U_1(0)}{U_1(0)} + \frac{\delta U_1(0)}{U_1(0)}$$

$$+ \frac{1}{\nabla T_m |x=0} \frac{d \delta T_m}{dx} |_0 + \frac{\delta H_{\theta}(0)}{\delta T_m |x=0} \frac{d \delta T_m}{dx} |_0$$

$$= \delta H_b(0) + \delta H_c(0) + \delta H_{\theta}(0) + \delta H_m(0),$$ \hspace{1cm} (20)

and similarly for $\delta H(x_w)$.

As long as $2 \delta U_1(0)/U_1(0)$ and $(d \delta T_m/\nabla T_m |x=0) \frac{d \delta T_m}{dx} |_0$ do not cancel or sum to a value much smaller than either individual term, $\delta H_b(0)$, $\delta H_c(0)$, and $\delta H_{\theta}(0)$ can be compared simply by comparing the magnitudes of $H_b$, $H_c$, and $H_{\theta}$. For $T_w < T_a$, both terms have the same sign. However, for $T_w > T_a$, they have opposite signs. In this case, Eqs. (5), (17), (29), and (37) can be used to show that the sum of these two terms is never small compared to the either of individual terms themselves.

To compare $H_c$ and $\tilde{H}_c$, $\tilde{H}_c$ is expanded in the limit ($r_h / \delta_e)^2 = (r_h / \delta_s)^2 \ll 1$, where $\delta_e$ and $\delta_s$ are the thermal and viscous penetration depths and $r_h$ is hydraulic radius.\textsuperscript{21} This expansion is equally valid at low Reynolds numbers in screen-based regenerators where $r_h$ is a rough measure of the average spacing between screen wires. In typical regenerators $(r_h / \delta_e)^2 \approx 0.1$. After a small amount of algebra, the result is found to be

$$\tilde{H}_c = \frac{\rho_n c_p U_l^2}{2 \sigma A} \frac{r_h}{\delta_e} \frac{d T_m}{dx}$$ \hspace{1cm} (21)

Taking the ratio of $\tilde{H}_c(0)$ with $H_c(0)$ yields

$$\frac{\tilde{H}_c(0)}{H_c(0)} = \frac{2(1 - \phi)}{\phi} \left[ \frac{\delta_e^2}{\xi_1(0) r_h} \right] \frac{2 k_s}{k},$$ \hspace{1cm} (22)
where $\xi_1(0)$ is the acoustic displacement amplitude and all terms are evaluated at the ambient end of the regenerator. Substituting typical numbers $k_j/k = 100$, $(\delta_\kappa/\xi_1)^2 \approx 10$, $\delta_\kappa/\xi_1 \approx 0.01$, and $\phi \approx 0.7$ yields $H_k/H_c \approx 0.1$. In a screen-bed regenerator, this ratio is expected to be even smaller due to the poor thermal contact between screen layers. Therefore, $\delta H_k(0) \ll \delta H_c(0)$ and can be neglected in the right-hand side of Eq. (20).

To compare $\delta H_b(0)$ with $\delta H_c(0)$, $H_b$ is expanded in the limit $(r_h/\delta_\kappa)^2 \rightarrow 0$,

$$H_b \approx \left( \frac{r_h}{\delta_\kappa} \right)^4 \frac{p_1 U_1}{3}.$$ (23)

Taking the ratio of $H_b$ with $H_c$ at the ambient end of the regenerator yields

$$\frac{H_b(0)}{H_c(0)} \approx 4\phi(\gamma - 1)\Gamma \left( \frac{r_h}{\delta_\kappa} \right)^2 \frac{T_a}{\Delta T_m}.$$ (24)

Here, $\gamma$ is the ratio of specific heats, $\lambda$ is the acoustic wavelength, $\Delta T_m$ is the total temperature difference across the regenerator, $\Gamma = (p_1/\lambda U_1)/(p_2 c_a/A)$, and $p_2$ and $c_a$ are the gas density and speed of sound at the ambient end. Substituting typical numbers $\phi = 0.7$, $\gamma - 1 \approx 0.67$, $(r_h/\delta_\kappa)^2 \approx 0.1$, $x_w/\lambda \approx 0.005$, $T_a/\Delta T_m \approx 1$, and $\Gamma \approx 10$ yields $|H_b(0)/H_c(0)| \approx 0.01$. Therefore, $\delta H_b(0) \ll \delta H_c(0)$ and can be neglected in the right-hand side of Eq. (20).

Finally, $\delta H_c(0)$ is compared with $\delta H_m(0)$. More specifically, $H_c(0)\delta U_1(0)/U_1(0)$ is compared with $\delta M_{2,0}(0) c_p T_a$. To estimate $\delta M_{2,0}(0)$, only one component is used, namely $\text{Re}[\rho_1(0) \delta U_1(0)]/2$. Assuming an isothermal density oscillation, the ratio of the two terms is

$$\frac{\delta M_{2,0}(0) c_p T_a}{H_c(0)\delta U_1(0)/U_1(0)} \approx 10\frac{\gamma \phi x_w}{\sigma \lambda} \left( \frac{r_h}{\delta_\kappa} \right)^2 \frac{T_a}{\Delta T_m} \Gamma \approx 10,$$ (25)

where the numerical value results from substituting typical numbers given previously. The approximation $\delta M_{2,0}(0) = \text{Re}[\rho_1(0) \delta U_1(0)]/2$ most likely overestimates $\delta M_{2,0}(0)$.

Now that $\delta H_c$ and $\delta H_m$ have been determined as the dominant terms in $\delta H$, it remains to calculate these terms at either end of the regenerator. At $x = 0$ and $x = x_w$, $\delta H_c$ is given by

$$\delta H_c(0) = H_c(0) \left[ \frac{2}{U_1(0)} \frac{\delta U_1(0)}{U_1(0)} + \frac{1}{\Delta T_m} \frac{d\delta T_m}{dx} \right].$$ (26)

$$\delta H_c(x_w) = H_c(x_w) \left[ \frac{2}{U_1(x_w)} \frac{\delta U_1(x_w)}{U_1(x_w)} + \frac{1}{\Delta T_m(x_w)} \frac{d\delta T_m}{dx} \right].$$ (27)

In the steady state, $H_c \gg H_b$, $H_c \gg H_k$ and $H_m \approx 0$. Therefore, $H_c(0) = H_c(x_w) = H_c$. Also, $\delta U_1(0)/U_1(0) = \delta U_1(x_w)/U_1(x_w)$. The difference in $\delta H_c$ across the regenerator, needed for Eq. (18), is then

$$\delta H_c(0) - \delta H_c(x_w) = H_c(x_w) \left[ \frac{2}{U_1(x_w)} \frac{\delta U_1(x_w)}{U_1(x_w)} + \frac{1}{\Delta T_m(x_w)} \frac{d\delta T_m}{dx} \right].$$ (28)

To be consistent with $H_c(0) = H_c(x_w)$, Eq. (21) shows that the mean temperature gradients at the two ends are related by

$$\Delta T_m(x_w) = \tau^{\beta} \Delta T_m(0),$$ (29)

where $\tau = T_w/T_a$. Using this in Eq. (28), the difference in $\delta H_c$ is found to be

$$\delta H_c(0) - \delta H_c(x_w) = H_c \frac{\pi}{\Delta T_m(x_w) (1 + \tau^{-\beta})}.$$ (30)

For both an FPTR and a TASHF, the term $H_c/\Delta T_m(0)$ is negative. Therefore, if the perturbation of the regenerator temperature is positive ($\Theta > 0$), the effect of $\delta H_c$ is to remove energy and cool the regenerator. This effect, present in any Stirling-type engine or refrigerator whether or not a toroidal topology allows streaming, attempts to reduce any perturbation from the steady state, suppressing a possible instability.

Next, the effect of $\delta H_m$ is determined. The difference in $\delta H_m$ at the two regenerator faces is given by

$$\delta H_m(0) - \delta H_m(x_w) = \delta M_{2,0}(0) c_p T_a - \delta M_{2,0}(x_w) c_p T_w.$$ (31)

To proceed further, a relationship between $\delta M_{2,0}(0)$ and $\delta M_{2,0}(x_w)$ must be determined. In the steady-state solution, the time-averaged continuity equation reveals that since $\partial \rho_m/\partial t = 0$, $M_{2,0}(0) = M_{2,0}(x_w)$. For $\delta M_{2,0}$, it is not clear if this relationship still holds. Let $\alpha = \delta M_{2,0}(0) - \delta M_{2,0}(x_w)$. If $\alpha \neq 0$, then the density of the gas in the regenerator must be changing, i.e., $\alpha = \phi A x_w \rho_m$. This change in density is driven by either a change in mean temperature or a change in mean pressure. Considering a change in mean temperature,

$$\alpha = \phi A x_w \frac{\rho_m}{T_m}.$$ (32)

However, for a $\delta T_m$ to change in the first place, there must be a $\delta M_{2,0}$ given by

$$(1 - \phi)\rho_c A x_w \epsilon \delta T_m = \delta M_{2,0} c_p (T_w - T_a).$$ (33)

Combining the last two equations yields

$$\frac{\alpha}{\delta M_{2,0}} \approx \frac{\phi \rho_m c_p}{(1 - \phi) \rho_c} \frac{T_w - T_a}{T_m} \ll 1,$$ (34)

showing that, to a good approximation, $\delta M_{2,0}(0) = \delta M_{2,0}(x_w)$. Therefore, the difference in $\delta H_m$ at the two ends of the regenerator can be written

$$\delta H_m(0) - \delta H_m(x_w) = \delta M_{2,0}(0) c_p T_a (1 - \tau).$$ (35)
Perturbing Eq. (7) and using \( \dot{M}_{2,0}=0 \) in the steady state, \( \delta U_1(0)/U_1(0) = -\delta R/R_m \) from Eq. (5), and \( \delta \rho \dot{p}_{2,0}/\rho \dot{p}_{2,0} = -2 \delta R/R_m \) from Eq. (9) yields

\[
\delta M_{2,0}(0) = -\frac{p_a}{p_m} \frac{p_1 U_1(0)}{2} \frac{\delta \rho \dot{p}_{2,0}}{\rho \dot{p}_{2,0}} = \frac{2 \rho_a E_2(0)}{p_m} \frac{\delta R}{R_m},
\]

where \( E_2(0) \) is the acoustic power flowing into the ambient end of the regenerator. \( \delta \rho \dot{a}/\rho \dot{a} \) is calculated by perturbing Eq. (6):

\[
\frac{\delta R}{R_m} = \frac{1}{b} \int_0^\infty T_m(x) \sin(\pi x/a) dx \frac{\Theta}{\Gamma} = \frac{\Theta}{\Gamma} f(\tau, b),
\]

where, to be consistent with Eq. (29), \( T_m(x) \) is approximated by

\[
T_m(x) = T_a \left[ 1 + \left( \frac{\tau - 1}{\pi} \right) \frac{\pi x}{a} - \frac{\pi x}{a} \right],
\]

and the variable of integration is changed to \( d(x/l/a) \). A plot of \( f(\tau, b) \) for various values of \( b \) is shown in Fig. 2. Finally, combining Eqs. (35), (36), and (37) yields

\[
\delta \dot{H}_m(0) = \frac{2 \gamma}{\gamma - 1} (1 - \tau) f(\tau, b) E_2(0) \frac{\Theta}{\Gamma}.
\]

From Eq. (39) it is clear what drives the instability. For \( \Theta > 0 \), the sign of the right-hand side of this expression is determined by \( (1 - \tau) \). All other factors are positive. For a TASHE, \( (1 - \tau) \) is negative and the perturbed time-average mass flux removes energy from the regenerator, reducing the original perturbation. For an FPTR, \( (1 - \tau) \) is positive and energy is dumped into the regenerator, amplifying the perturbation. Equation (30) already showed that no matter the value of \( \tau \), \( \delta \dot{H}_m \) reduces the original perturbation. For \( \tau > 1 \), i.e., a TASHE, a jet pump can reliably be used to suppress streaming. For \( \tau < 1 \), i.e., an FPTR, there exists a possibility of an instability with \( \delta \dot{H}_c \) competing with \( \delta \dot{H}_m \).

Now that the term that drives the instability is identified, the cause can be investigated. In either an FPTR or a TASHE, if \( \Theta \) is positive, the average temperature of the regenerator increases slightly and its resistance grows, reducing \( U_1(0) \). The effect on the two terms in Eq. (7) is slightly different. The smaller \( U_1(0) \) results in a lower \( \text{Re} \{ \rho_1(0) \dot{U}_1(0) \} \), but \( |U_{2,0}| \) decreases even more because it varies as \( U_1^2(0) \) through its dependence on \( \Delta \rho \dot{p}_{2,0} \). Therefore, when the regenerator warms slightly, \( M_{2,0} \) increases slightly, i.e., \( \delta M_{2,0}(0) > 0 \). In a TASHE, this blows ambient gas into the regenerator, cooling it down and suppressing any instability. In an FPTR, the ambient gas warms the regenerator further, creating an instability.

Combining Eqs. (18), (30), and (39) gives an equation for the growth rate of the perturbation

\[
\frac{2}{\gamma} \rho_a c_x T_a (1 - \phi) A_{x_u} = \frac{\pi (T_a/x_u)}{\nabla T_m |_{0}} (1 + \tau^{-b}) \dot{H}_c + \frac{2 \gamma}{\gamma - 1} (1 - \tau) f(\tau, b) E_2(0).
\]

If the right-hand side of this equation is positive, the perturbation will grow exponentially and an instability results with a large streaming mass flux around the lumped-element loop. If it is negative, the perturbation decays and the jet pump controls the streaming in a stable manner. As already discussed, both terms on the right-hand side are negative for a TASHE. Therefore, a jet pump can be used in a TASHE with confidence that it will always control the streaming. In an FPTR, the first term is negative and the second is positive. Therefore, they must be compared to see when an instability results.

To make this comparison more transparent, \( E_2(0) \) is expressed in terms of the gross cooling power of the FPTR, i.e., \( \dot{Q}_{c,\text{gross}} = \tau E_2(0) \).

\[
\dot{Q}_{c,\text{gross}} < \frac{2}{\gamma} \frac{\gamma - 1}{\gamma - 1} \frac{(1 - \tau) f(\tau, b)}{(1 + \tau^{-b})} \frac{\nabla T_m |_{0}}{(T_a/x_u)}.
\]

Because \( \nabla T_m |_{x_u} = \tau^b \nabla T_m |_{0} \), the term \( \nabla T_m |_{0} / (T_a/x_u) \) is given by \( 2(\tau - 1)/(1 + \tau^b) \). Substituting this result into Eq. (41), the final result is

\[
\frac{\dot{H}_c}{\dot{Q}_{c,\text{gross}}} < 4 \frac{\gamma - 1}{\gamma - 1} \frac{(1 + \tau^{b})(1 + \tau^{-b})}{(1 + \tau^{-b})}.
\]

The right-hand side of Eq. (42) is computed as a function of \( \tau \) for \( \gamma = 5/3 \) and \( b = 0, b = 0.68 \) (typical for helium), and \( b = 0.85 \) (typical for argon). The results are shown in Fig. 3. For FPTRs operating below and to the left of the lines, a jet pump cannot be used to control streaming in a stable fashion. For FPTRs operating above and to the right of the lines, a jet pump will suppress streaming in a stable fashion. The open squares in Fig. 3 are two different operating points from a ≈ 2-kW FPTR intended to liquefy natural gas. The jet pump in this FPTR never was able to reliably control the streaming. Since this FPTR used helium as its working gas, the squares should be compared to the \( b = 0.68 \) line. The circles are from an earlier benchtop FPTR. The filled circles
FIG. 3. Stability curve. The lines are the threshold of instability when the left-hand side of Eq. (42) equals the right-hand side for \( b=0 \), \( b=0.68 \), and \( b=0.85 \). As shown by Eqs. (22) and (24), \( H_1 \) and \( H_b \) are much smaller than \( H_t \) so that \( H_2=H_t \). Below and left of these lines, a jet pump will not suppress streaming in a stable manner. The open squares left of the line are data from an FPTR that demonstrated a streaming instability (Ref. 11) and should be compared with the \( b=0.68 \) line. The circles are from an earlier benchtop FPTR (Ref. 5). The filled circles are operating points where the jet pump controlled the streaming stably. The open circle is an operating point where the streaming control was unstable. The circles should be compared to the \( b=0.85 \) line. In both cases, \( H_2 \) and \( Q_{\text{gross}} \) are not directly measured. They are calculated using DeltaE (Ref. 24).

are operating points where the streaming control was stable. The open circle is an operating point with unstable streaming control. This FPTR used argon as a working gas, so the circles should be compared with the \( b=0.85 \) line.

To be consistent with the steady-state condition \( \dot{H}_c(0) = \dot{H}_c(x_w) \) and the neglect of regenerator compliance in Eq. (5), the approximate mean temperature given in Eq. (38) was used throughout the calculation. However, in actual hardware where the compliance of the regenerator may be important or the phase of \( U_1(0) \) relative to \( p_1 \) may be significantly different from zero, \( T_m(x) \) may deviate from the temperature profile used in the calculation. The calculation has been redone for a linear temperature profile, i.e., a constant \( \nabla T_m(x) \). There are differences with the results shown in Fig. 3, but they are small enough not to be noticeable on the plot. The result that a TASHE is always stable still holds for a linear temperature profile as well.

IV. CONCLUSION

Using a simplified model of the acoustics in the regenerator of a TASHE and FPTR, the stability of jet pump control of streaming has been investigated. A stability criterion has been derived and found to be in agreement with the meager data available to date. The stability criterion shows that jet pump control of streaming is stable for all TASHEs. It also provides a threshold temperature ratio below which streaming control in a FPTR is unstable. The mathematics is based on analyzing how the temperature in the center of the regenerator responds to changes in streaming flows through the regenerator, which are themselves controlled by the temperature in the center of the regenerator. The analysis shows that two effects dominate.

First, ordinary second-order energy flow through the regenerator, whose largest term is proportional to \( U_1^2 dT_m/dx \), always exerts a stabilizing influence. If the temperature of the center of the regenerator of a TASHE decreases a small amount, enthalpy flow from the hot end to the center increases as the average temperature gradient in the hot half steepens, and enthalpy flow from the center to the ambient end decreases as the average temperature in the cooler half becomes more shallow; both of these changes in enthalpy flow tend to raise the temperature in the center, canceling the original, assumed decrease in temperature. Similar arguments for an assumed small increase in the center temperature, and for an FPTR instead of a TASHE, also lead to cancellation, and, hence, stability.

Second, the temperature dependences of the viscosity and density of the gas in the regenerator cause a change in streaming that affects that very temperature. In an FPTR, this effect is destabilizing. If the temperature of the center of the regenerator of an FPTR decreases a small amount, the viscosity decreases and the density increases; both of these changes reduce \( R \), leading to increases in both \( U_1 \) and \( U_{2,0} \). If \( \Delta p_{2,0} \) exerted by the jet pump were to remain constant, the fractional changes in \( U_1 \) and \( U_{2,0} \) would be equal to the fractional change in \( R \), and the streaming—a balance between \( U_1 \) and \( U_{2,0} \)—would change little. However, \( \Delta p_{2,0} \) does not remain constant; it increases, thereby changing the streaming in a direction that carries cold gas into the regenerator, amplifying the original temperature decrease. A similar argument for a TASHE leads instead to stability.

Operation of a cryogenic FPTR with deliberately large \( H_2 \) to enforce stability is very undesirable, because nonzero \( H_2 \) consumes cooling power, reducing efficiency. However, the present analysis hints at ways that the stability curve might be shifted slightly. Three examples will be mentioned. First, the analysis assumed that \( K_{\text{out}} \), \( K_{\text{in}} \), and \( A_{\text{jp}} \) in Eq. (9) are independent of amplitude. If one or more of these coefficients depended on \( U_1 \), either via hydrodynamics or elastic motion of the jet pump walls, a region of enhanced stability could be created. Second, Eq. (42) shows that reduced \( \gamma \) or increased \( b \) improves stability. Third, the analysis assumed that \( R \), \( f_s \), and \( f_v \) are independent of velocity, but the more complicated, velocity-dependent flow resistance and heat transfer coefficient in screen beds may provide an opportunity for improved stability.

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Unpublished data from 2000 collaboration between Chart, Inc. and Los Alamos National Laboratory.

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G. W. Swift, Thermoacoustics: A Unifying Perspective for some Engines and Refrigerators (Acoustical Society of America, NY, 2002).

In regenerators, typical Reynolds numbers are on the order of 10 to 100. For parallel plates and circular or rectangular pores, the low-Reynolds number limit applies up to Reynolds numbers of ~1000 or higher (Ref. 23) where a relatively sharp transition from laminar to turbulent flow is observed. However, the transition is not so sharp in screen beds. In this case, the flow can be characterized by a “laminar” flow resistance, i.e., one that does not depend on Reynolds number, and a turbulent flow resistance that increases linearly with Reynolds number (Ref. 19). For screen bed porosities in the range 0.65 to 0.75, the laminar and turbulent contributions are equal for Reynolds numbers of 90 to 130. This is the upper range of Reynolds numbers in regenerators. The data presented in this article are taken at Reynolds numbers of ~20 (Ref. 5) and ~45 (Ref. 11).


Waxler (Ref. 17) shows that \( dp_{20}/dx \) dominates other contributions to \( U_{20} \) by a factor of order \( (\delta_x/r_h)^2 \), where \( \delta_x \) is the thermal penetration depth and \( 2r_h \) is the gap in the parallel-plate regenerator. This factor is always large for regenerators that are at least reasonably effective. The neglect of the temperature dependence of viscosity is not always acceptable in streaming calculations [see, e.g., J. R. Olson and G. W. Swift, “Acoustic streaming in pulse tube refrigerators: Tapered pulse tubes,” Cryogenics 37, 769–776 (1997)]. Incorporating this effect into Waxler’s analysis adds a term \( (\partial^2/\partial y \partial x) \mu_1 \partial^2/\partial y^2 \) to his Eq. (21), in his notation. This term also turns out to be only of order \( (r_h/\delta_x)^2 \).


The hydraulic radius is the ratio of the gas volume to the gas–solid contact area. For example, for parallel plates \( r_h \) is half of the distance between the plates.

