Improved predictive sampling using quasi-Monte Carlo with application to neutron-cross-section evaluation

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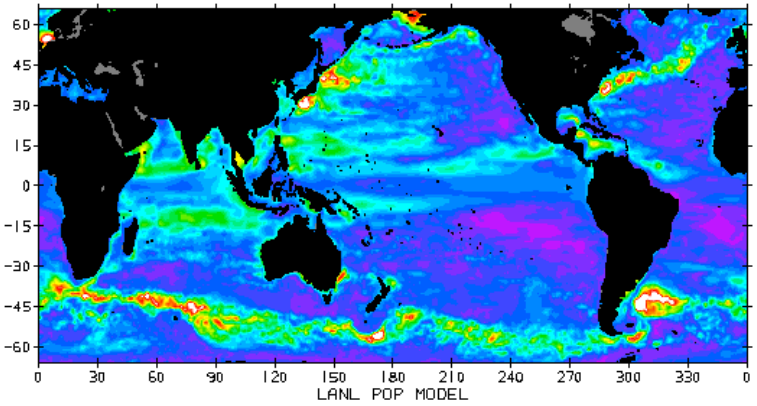
Overview

- Quasi-Monte Carlo (QMC) – purpose
- Digital halftoning – purpose and constraints
- New approaches to generating sample sets with uniform spacing
  - halftoning algorithm provides good point sets for QMC
  - leads to Repulsive Particle Model (RPM) and Centroidal Voronoi Tessellation (CVT)
- Point sets for sampling distributions in high dimensions
  - predictive sampling – estimate prediction mean and uncertainty
- Neutron cross-section evaluation
  - combine directly measured neutron cross sections for fission of ^{239}Pu with a high-accuracy criticality measurement
  - Bayesian update
  - objective – to characterize prediction uncertainty using 30 samples for 30 input parameters

Context – big physics simulation codes

- Computer simulation codes
  - many input parameters, many output variables
  - very expensive to run; days to weeks on super computers
- Important to assess uncertainties in predictions - thus need to
  - compare codes to experimental data; make inferences
  - use advanced methods to estimate sensitivity of simulation outputs to inputs
    - Latin square (hypercube), stratified sampling, quasi-Monte Carlo, CVT
- Examples of complex simulations
  - nuclear-reactor design
  - ocean and atmosphere modeling
  - aircraft design; space shuttle design
  - casting of metals

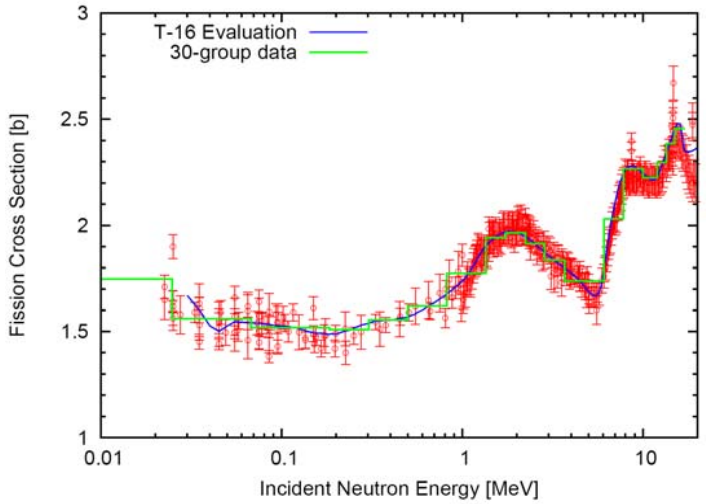
calculations can take 3 months!



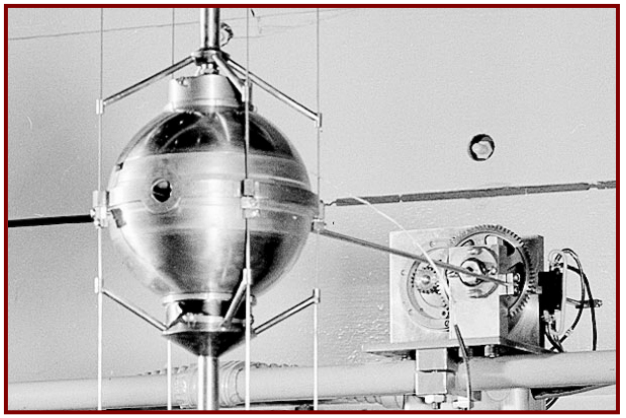
Our application

- Focus on neutron cross sections
- Aim to improve our knowledge of the cross sections by incorporating a high-precision criticality measurement; integral constraint
- Criticality experiment simulated with a discrete-ordinate code, based on 30 energy bins
- Ultimate goal is to use the improved cross sections to predict other similar physical situations
- Need to characterize prediction uncertainties; 30-D parameter space

Neutron fission cross section



Criticality experiment



Monte Carlo integration techniques

- Generic purpose of Monte Carlo
  - estimate integral of a function over a specified region R in m dimensions, based on evaluations at n sample points

$$\int_R f(\boldsymbol{x})\, d\boldsymbol{x} = \frac{V_R}{n} \sum_{\boldsymbol{i}=1}^{\boldsymbol{n}} f(\boldsymbol{x_i})$$

- Constraints
  - integrand not available in analytic form, but calculable
  - function evaluations may be expensive, so minimize number
- Algorithmic approaches – want best accuracy with fixed number of function evaluations n
  - simple quadrature (Riemann) – good for few dimensions; rms err ~ n^{-1}
  - Monte Carlo – useful for many dimensions; rms err ~ $n^{-1/2}$
  - quasi-Monte Carlo – reduce number of evaluations; rms err ~ n^{-1}

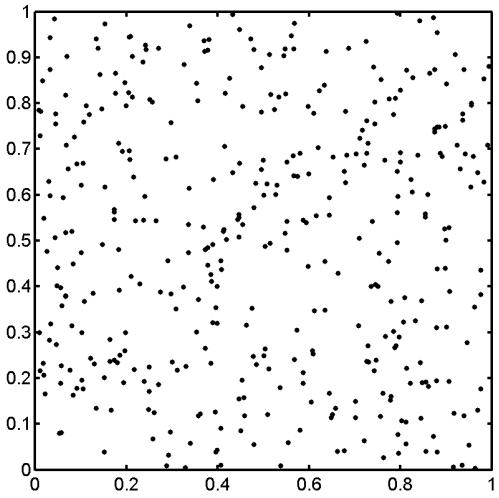
Quasi-Monte Carlo

- Purpose
  - estimate integral of a function over a specified domain in *d* dimensions
  - obtain better rate of convergence of integral estimation than occurs in classic Monte Carlo
- Constraints
  - integrand function not available analytically, but calculable
  - function known (or assumed) to be reasonably well behaved, e.g. smooth
- Standard QMC approaches use low-discrepancy sequences; product space (Halton, Sobel, Faure, Hammersley, …)
  - most studies usually involve many samples in a few dimensions
- Propose here new ways of generating sample point sets
  - our focus is on a few samples in high dimensions

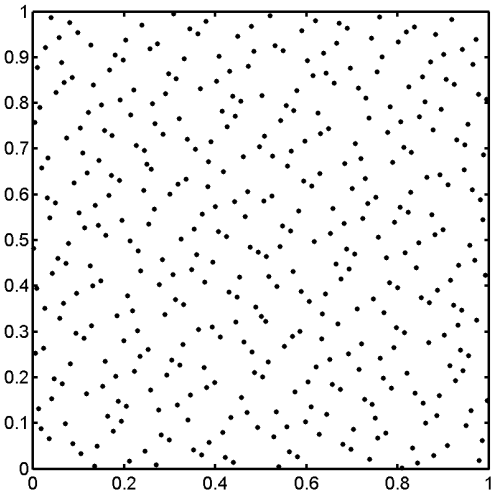
Point set examples

- Scatter plots of different kinds of point sets (400 points)
- Halton sequence reduces clustering that occurs in random seqs.
- If quasi-MC sequences have better integration properties than random, is halftone pattern even better?

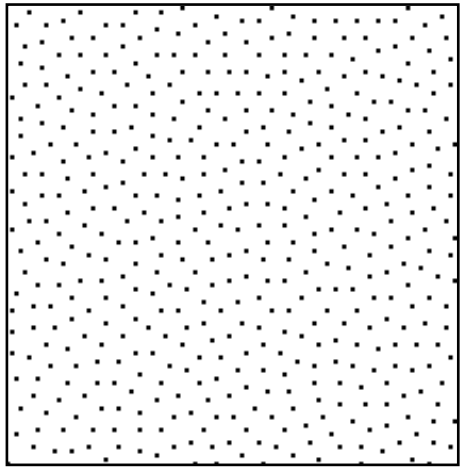
Random
(independent)



Quasi-Random
(Halton sequence)



Halftone
(DBS sky)

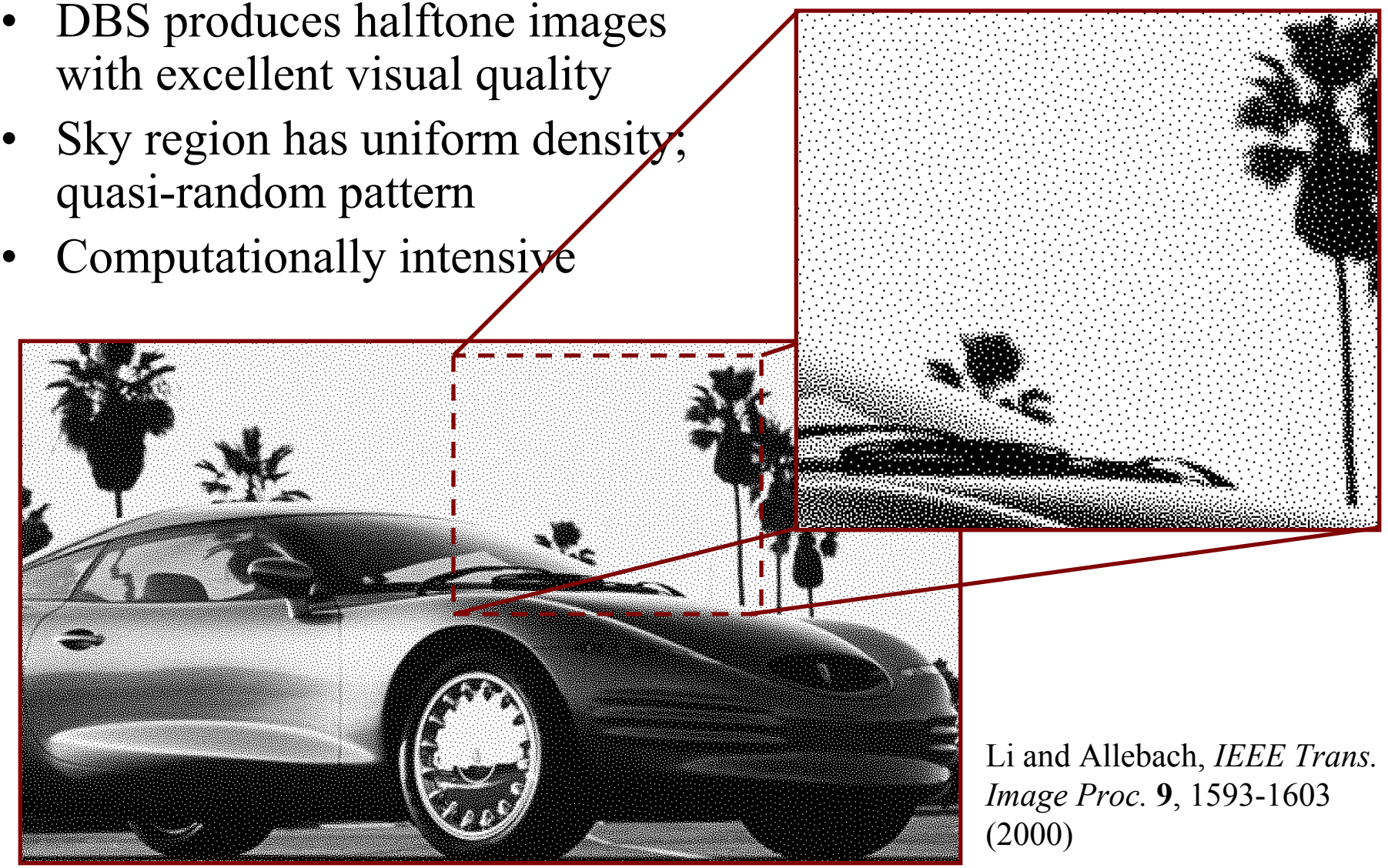


Digital halftoning techniques

- Purpose
  - render a gray-scale image by placing black dots on white background
  - make halftone rendering **look** like original gray-scale image
  - related to characteristics of human observer
  - important for laser printers
- Constraints
  - resolution – size and closeness of dots, number of dots
  - speed of rendering
- Various algorithmic approaches
  - error diffusion, look-up tables, blue-noise, …
  - concentrate here on Direct Binary Search (Allebach et al.)

Direct Binary Search example

- DBS produces halftone images with excellent visual quality
- Sky region has uniform density; quasi-random pattern
- Computationally intensive



Li and Allebach, *IEEE Trans. Image Proc.* **9**, 1593-1603 (2000)

Direct Binary Search (DBS) algorithm

- Consider digital halftone image to be composed of black or white pixels
- Cost function is based on perception of two images

$$\varphi = \left|\mathbf{h} * (\mathbf{d} - \mathbf{g})\right|^2$$

  - where **d** is the dot image, **g** is the gray-scale image to be rendered, * represents convolution, and **h** is the image of the blur function of the human eye, for example, $(w^2 + r^2)^{-3/2}$
- To minimize φ
  - start with a collection of dots with average local density ~ **g**
  - iterate sequentially through all image pixels;
  - for each pixel, swap value with neighborhood pixels, or toggle its value to reduce φ
- Edge effects must be dealt with
  - in above, dot image surrounded by field of uniform density

Minimum Visual Discrepancy (MVD) algorithm

Inspired by Direct Binary Search halftoning algorithm:

- Start with an initial set of points
- Goal is to create uniformly distributed set of points
- Cost function is variance in blurred point image

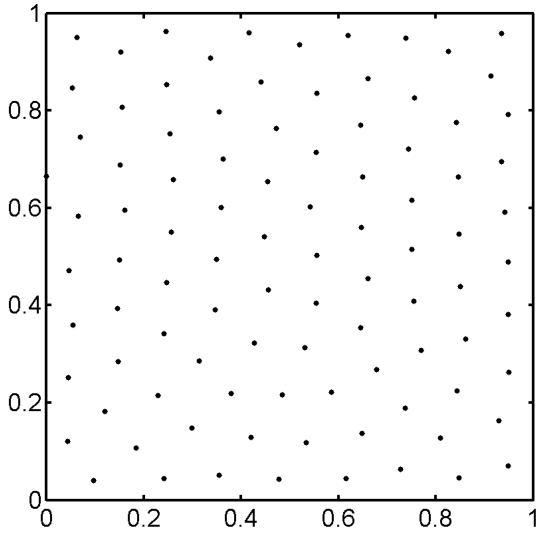
$$\psi = \mathrm{var}(\mathbf{h} * \mathbf{d})$$

  - where **d** is the point (dot) image, **h** is the blur function of the human eye, and * represents convolution
- Minimize ψ by
  - starting with some point set (random, stratified, Halton,…)
  - visiting each point in random order;
  - moving each point in 8 directions, and accept move that lowers ψ the most

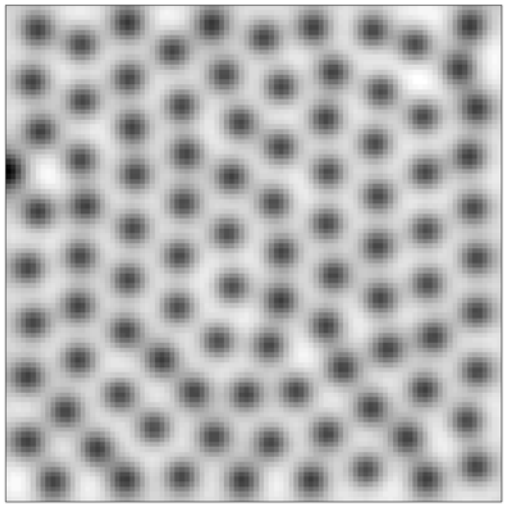
Minimum Visual Discrepancy (MVD) algorithm

- MVD result; start with 95 points from Halton sequence
- MVD objective is to minimize variance in blurred image
- Effect is to force points to be evenly distributed, or as far apart from each other as possible
- Might expect global minimum is a regular pattern

MVD, 95



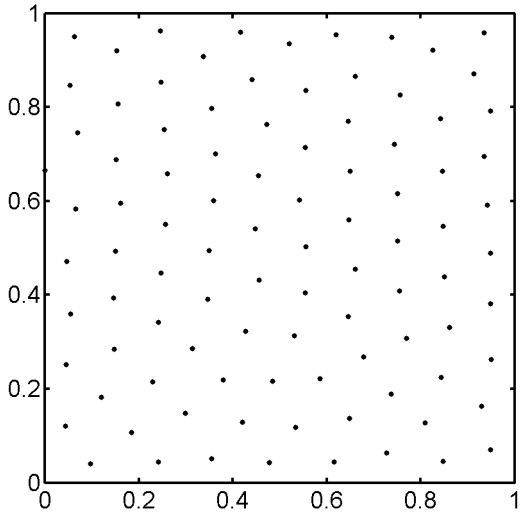
Blurred image



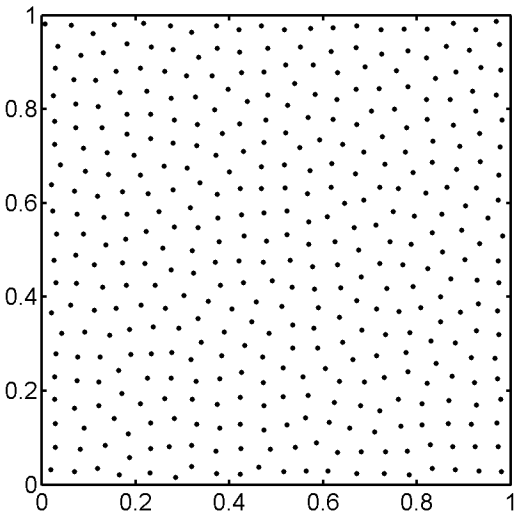
MVD results

- In each optimization, final pattern depends on initial point set
  - algorithm seeks local minimum, not global (as does DBS)
- Patterns somewhat resemble regular hexagonal array
  - similar to lattice structure in crystals or glass
  - however, lack long-range (coarse scale) order
  - best to start with point set with good long-range uniformity

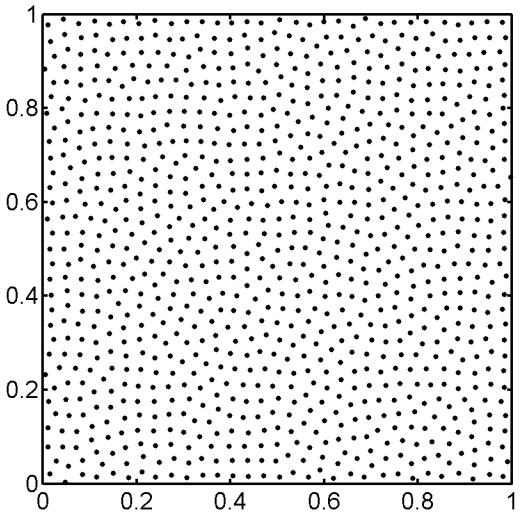
MVD, 95



MVD, 400



MVD, 1000



Repulsive particle model

- Model points as set of interacting (repulsive) particles
- Cost function is the potential

$$\psi = \sum_{i,\, j \geq i+1} V(\boldsymbol{x}_i, \boldsymbol{x}_j) + \sum_i U(\boldsymbol{x}_i)$$

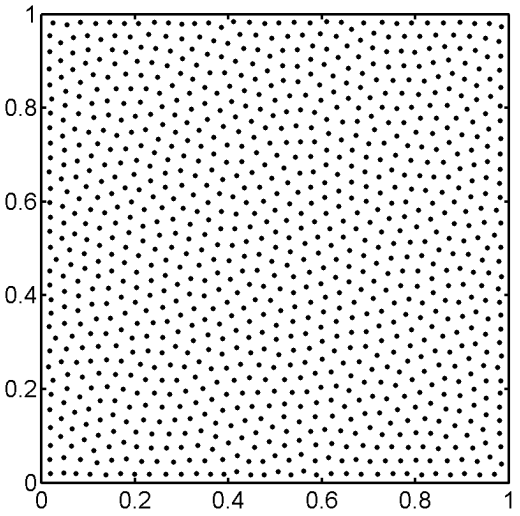
  - where V is a particle-particle interaction potential and U is a particle-boundary potential
  - particles are repelled by each other and from boundary
- Minimize ψ by moving particles by small steps
- This model is **analytically equivalent** to Minimum Visual Discrepancy (V and U directly related to blur function **h**)
- Suitable for generating point sets in high dimensions

Repulsive particle model

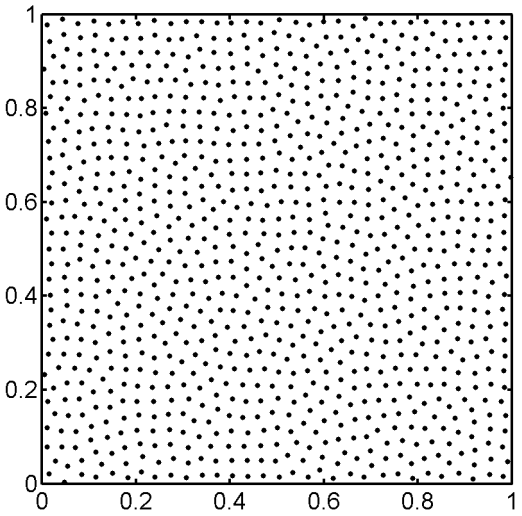
- Equivalent to Minimum Visual Discrepancy algorithm
- Example of repulsive-particle results
  - resulting point pattern is visually indistinguishable from MVD pattern

1000 points

Repulsive Particle Model



MVD

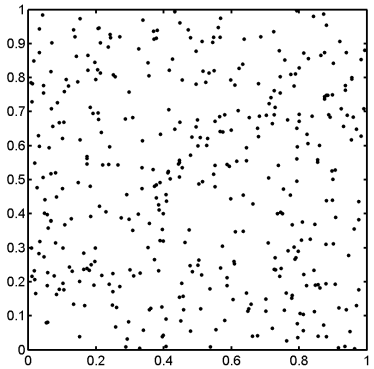


Point set examples

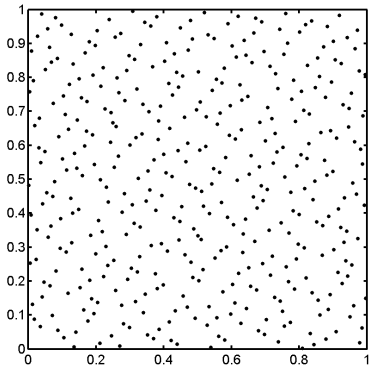
- Compare various kinds of point sets (400 points)
  - varying degrees of randomness and uniformity
- As the points become more uniformly distributed, the more accurate are the values of estimated integrals
- Example:

RMS relative accuracies of integral of $\text{func2} = \prod_{i} \exp\left(-2\left|x_i - x_i^0\right|\right); \quad 0 < x_i^0 < 1$

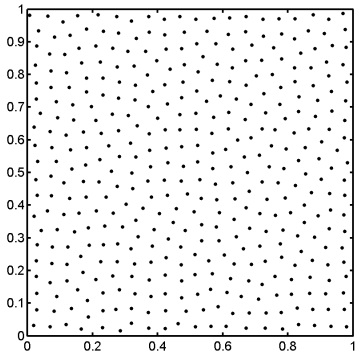
Random, 2.5%



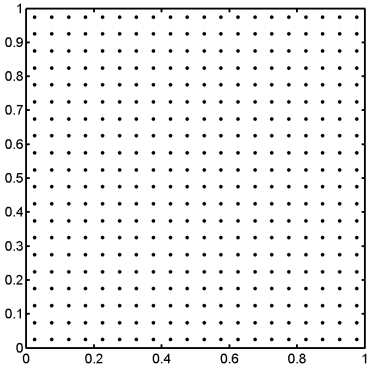
Halton, 0.5%



MVD, 0.14%

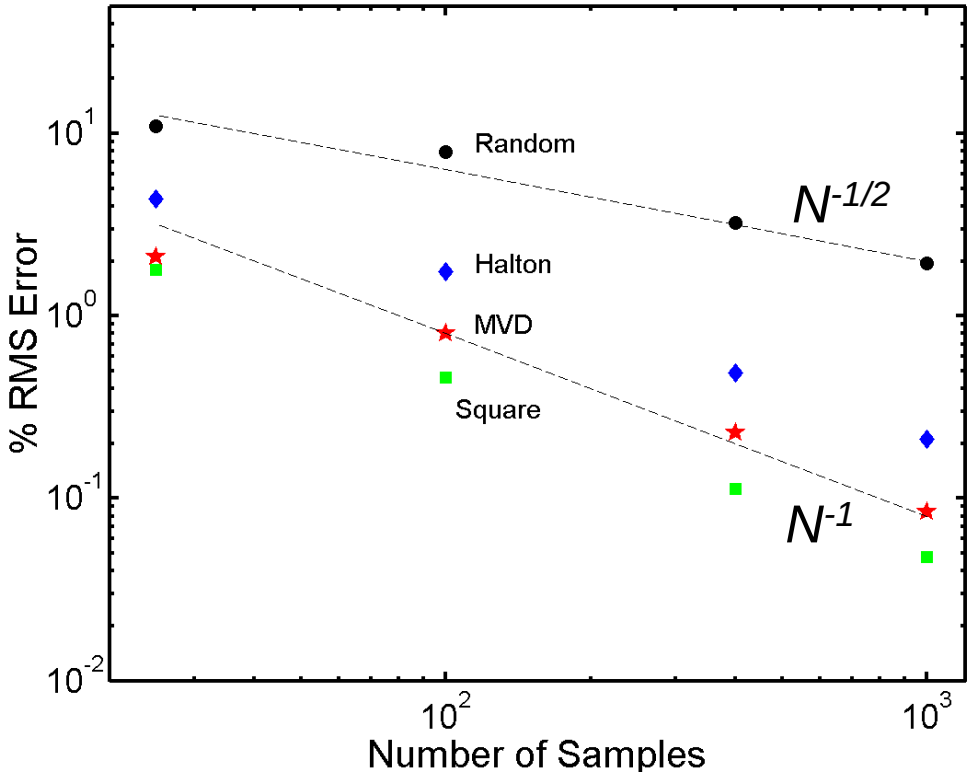


Grid, 0.09%



More Uniform, Higher Accuracy

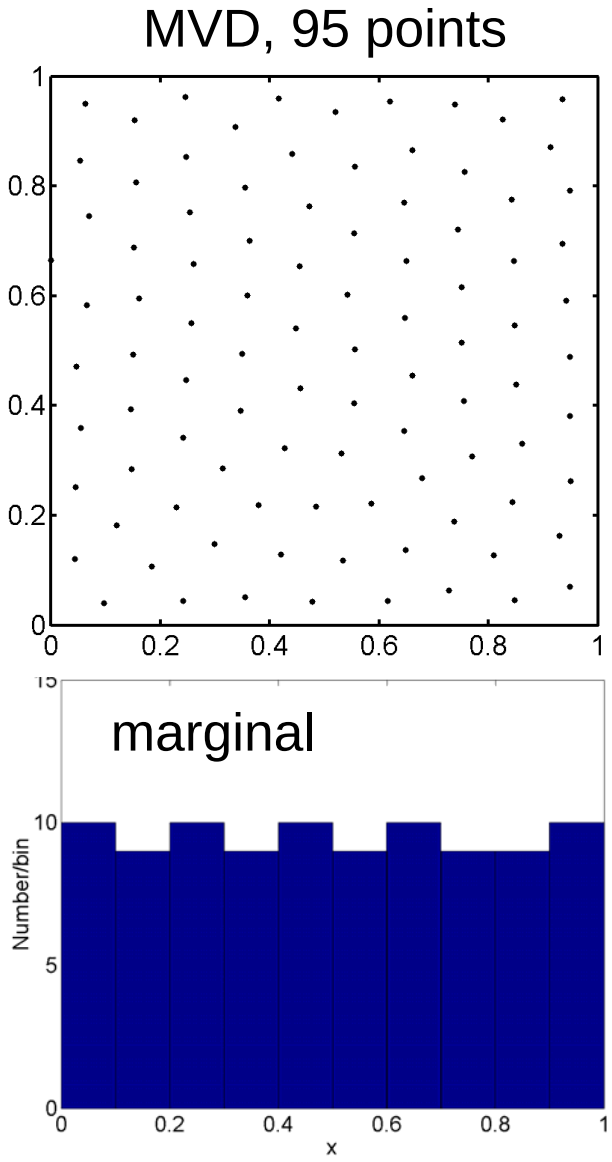
Integration test problem



- RMS error for integral of $\text{func2} = \prod_i \exp\left(-2\left|x_i - x_i^0\right|\right); \quad 0 < x_i^0 < 1$
  - integrate over x_i^0 by using MC, drawing x_i^0 from uniform distribution
  - from worst to best: random, Halton, MVD, square grid
  - lines show $N^{-1/2}$ (expected for MC) and N^{-1} (expected for QMC)

Marginals for MVD points

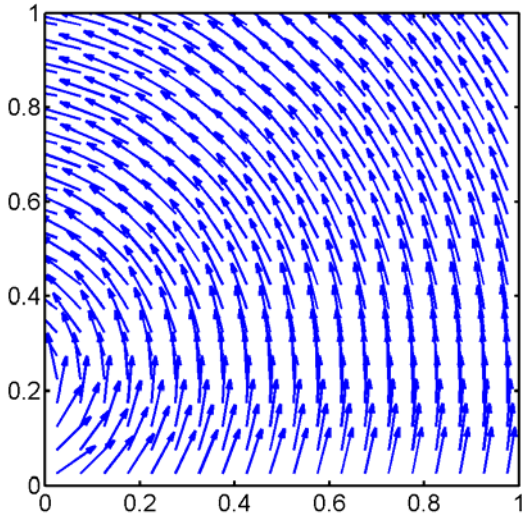
- Sometimes desirable for projections of high dimensional point sets to sample each parameter uniformly
- Latin hypercube sampling designed to achieve this property (for specified number of points)
- Plot shows histogram of 95 MVD samples along x-axis, i.e., marginalized over y direction
- MVD points have relatively uniform marginal distributions



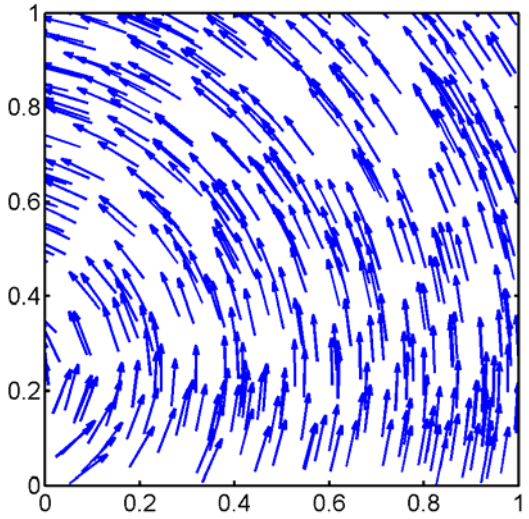
Another use of MVD: visualization of flow field

- Fluid flow often visualized as field of vectors
- Location of vector bases may be chosen as
  - square grid (typical) - regular pattern produces visual artifacts
  - random points - fewer artifacts, but nonuniform placement
  - quasi-random - fewest artifacts and uniform placement

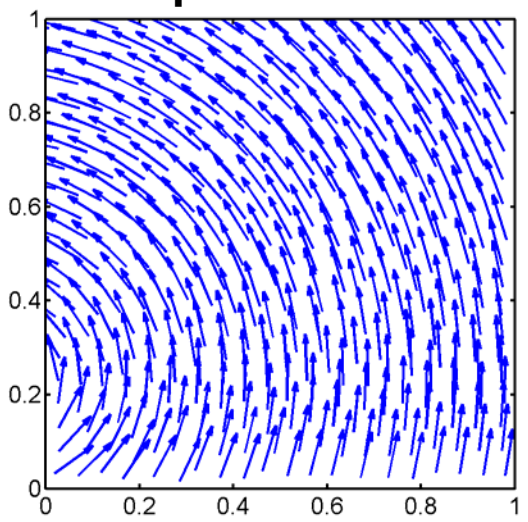
Square grid



Random points



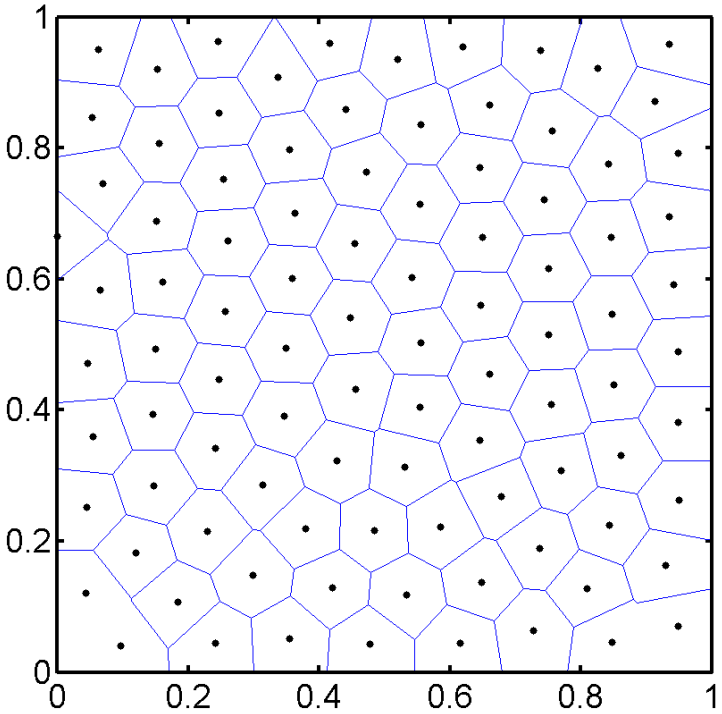
Quasi-random (MVD) point set



Voronoi analysis

- Voronoi diagram (2D)
  - partitions domain into polygons
  - points in *i*th polygon are closest to *i*th generating point, $\boldsymbol{x}_i$
  - boundaries shown are obtained by geometrical construction
- MC technique for Voronoi analysis
  - randomly throw large number of points $\boldsymbol{z}_k$ into region
  - compute distance of each $\boldsymbol{z}_k$ to all generating points $\{\boldsymbol{x}_i\}$
  - $\boldsymbol{z}_k$ belongs to Voronoi region of closest $\boldsymbol{x}_j$
  - can compute A_i, radial moments, identify neighbors, …
- Readily extensible to high dimensions

Voronoi analysis: MVD, 95



Voronoi analysis can improve classic MC

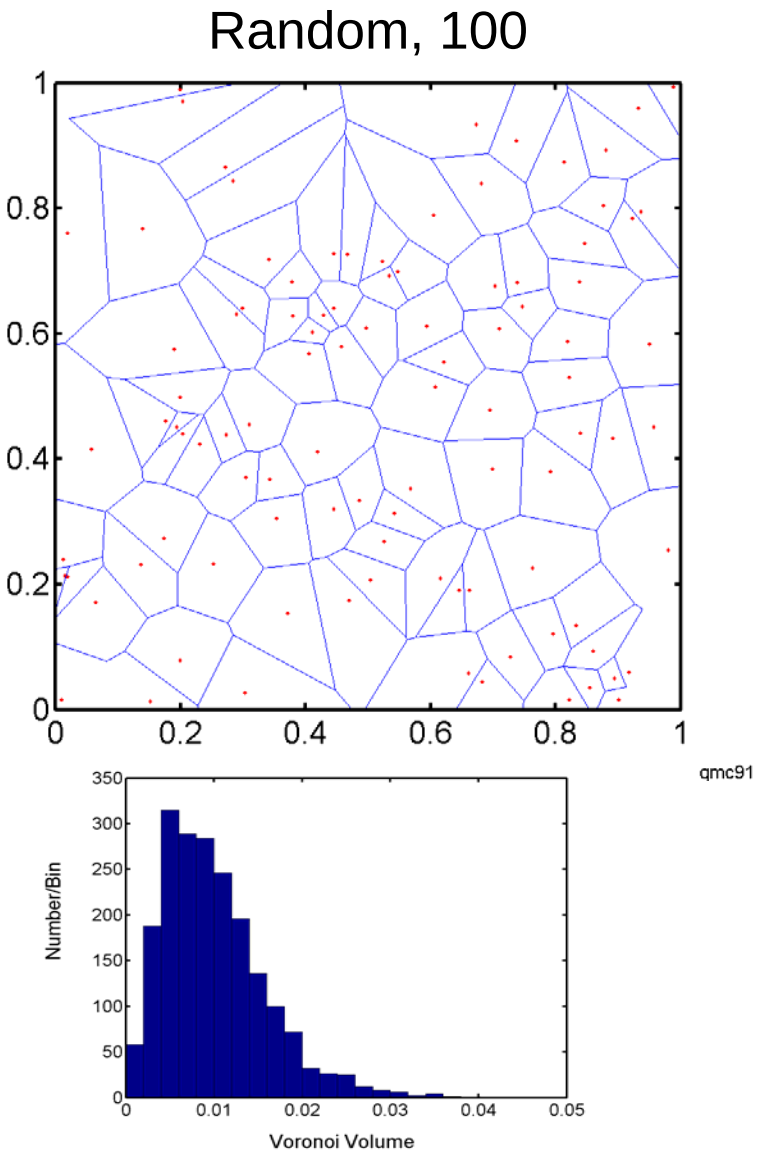
- Standard MC formula

$$\int_R f(\boldsymbol{x})\,d\boldsymbol{x} = \frac{V_R}{n}\sum_{i=1}^{n} f(\boldsymbol{x}_i)$$

- Instead, use weighted average

$$\int_R f(\boldsymbol{x})\,d\boldsymbol{x} = \sum_{i=1}^{n} f(\boldsymbol{x}_i)V_i$$

  - where V_i is the volume of Voronoi region for ith point; Riemann integr.
- Accuracy of integral estimate dramatically improved in 2D:
  - factor of 6.3 for $N = 100$ (func2)
  - factor of > 20 for $N = 1000$ (func2)
- Suitable for adaptive sampling
- Less useful in high dimensions (?)



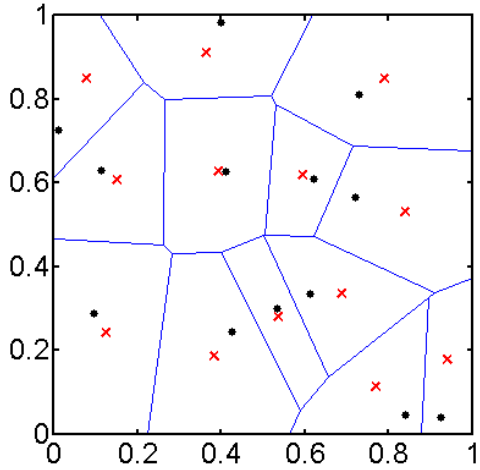
Centroidal Voronoi Tessellation

- Plot shows 13 random points (·) and the centroids of their Voronoi regions (×)
- A point set is called a Centroidal Voronoi Tessellation (CVT) when the generating points $\mathbf{z}^j$ coincide with the centroids their Voronoi regions; a CVT minimizes

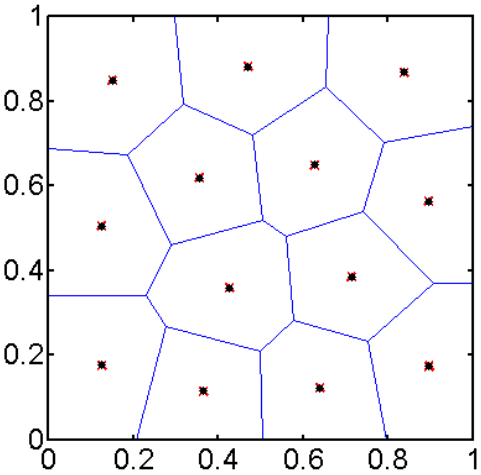
$$\sum_j \int_{V_j} \left|\mathbf{z}^j - \mathbf{x}\right|^2 d\mathbf{x}$$

- Algorithm (McQueen)
  - start with arbitrary set of generating points
  - perform Voronoi analysis using MC algorithm
  - move each generating point to its Voronoi centroid
  - iterate lasts two steps until convergence
- Final CVT points uniformly distributed

Start with random points



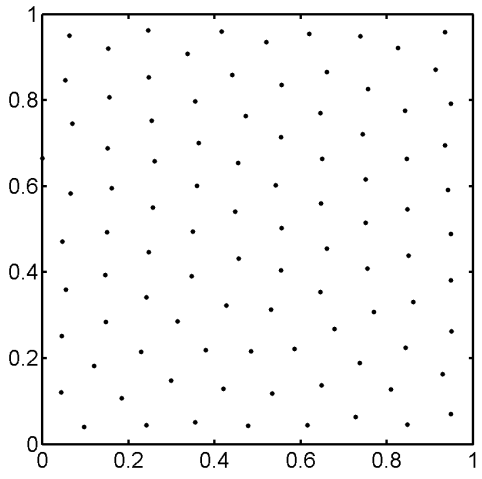
Final CVT point set



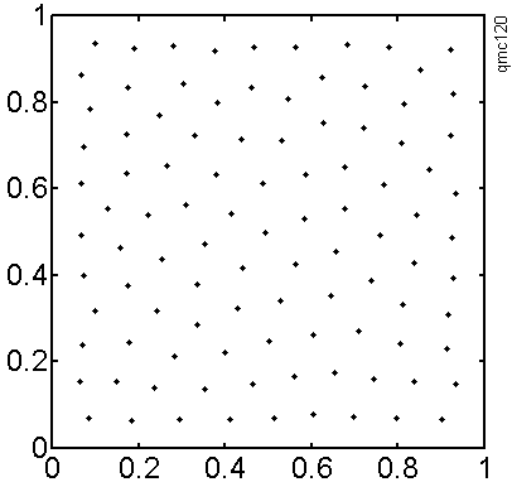
Comparison of methods

- Preceding three algorithms provide uniformly-space points, have essentially equivalent patterns, and are useful for QMC
  - Minimum Visual Discrepancy (MVD)
  - Repulsive Particle Model (RPM)
  - Centroidal Voronoi Tessellation (CVT)
- For high dimensions: both CVT and RPM may be useful, RPM likely most efficient

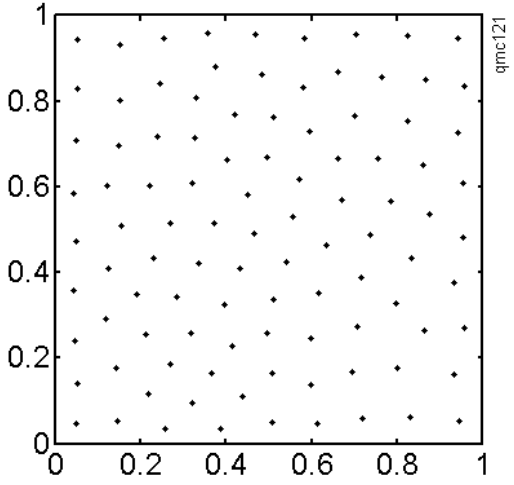
MVD, 95



RPM, 100

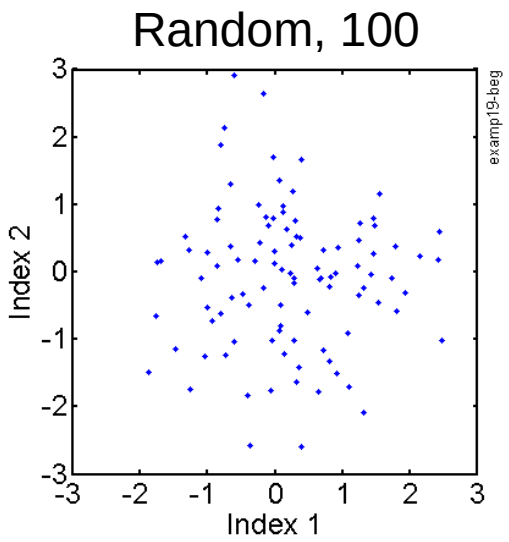


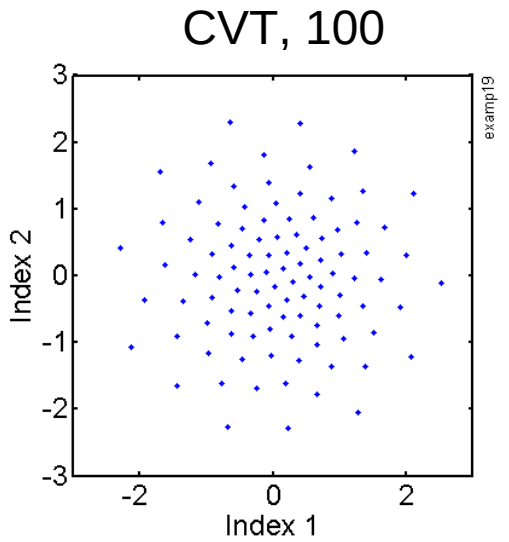
CVT, 100



CVT for multi-variate normal distribution

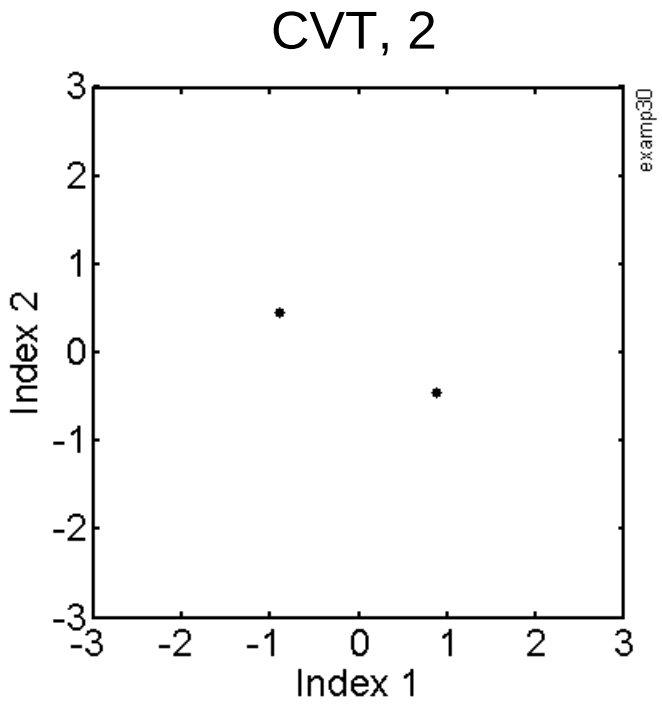
- CVT algorithm works for an arbitrary density function, e.g., a normal distribution
- In above MC algorithm for Voronoi analysis, simply draw random numbers from desired distribution
- Plots show starting random point set and final CVT set
- Radii of points are rescaled to achieve desired average variance along axes





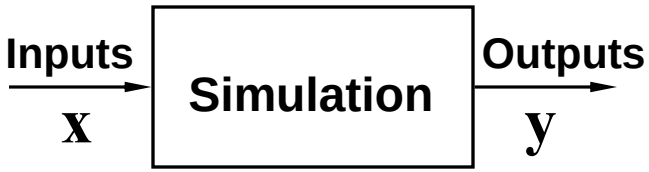
CVT: 2 points in 2 dimensions

- Bi-variate normal distribution is rotationally symmetric
- Symmetry of situation means that the CVT points must be symmetric about origin; at the same radius
- This pattern is unique,
  up to a random rotation
- Both x_1 and x_2 values sampled (with near certainty), but there is a subspace, orthogonal to x_1-x_2 line, whose dependence is not sampled
- Generalizing, the d-D space is under-sampled when $n < d + 1$



Recall context

- Our interest is in characterizing the uncertainty in a simulation output, based on what we know about the uncertainties in the inputs

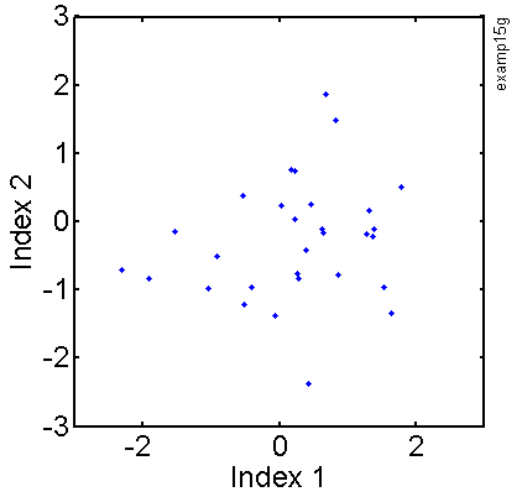


- But, the high cost of running the simulation limits how many samples can be drawn from a parameter distribution to obtain a predictive distribution
- We are often in a situation where the number of points is comparable to number of parameters $(n \approx d)$
- Our goal is to draw a modest number of points from a high-dimensional normal distribution
- Let's explore some of the characteristics of the problem by starting with the example of 2 sample points in 2D

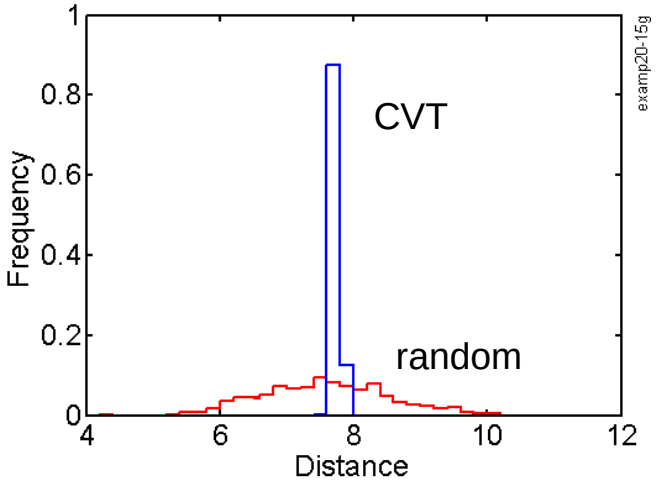
CVT: 30 points in 30 dimensions

- 30 dimensional normal distribution
- Projected onto 2D plane, CVT result doesn't look much different than random sample set
- However, CVT points are uniformly separated in *d*-D, while random points are not

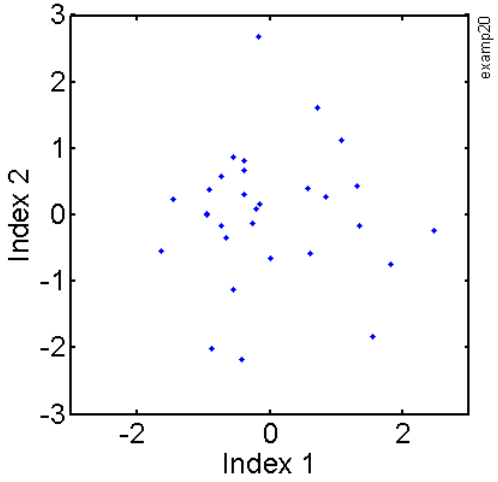
Random, 30



Point separation histogram



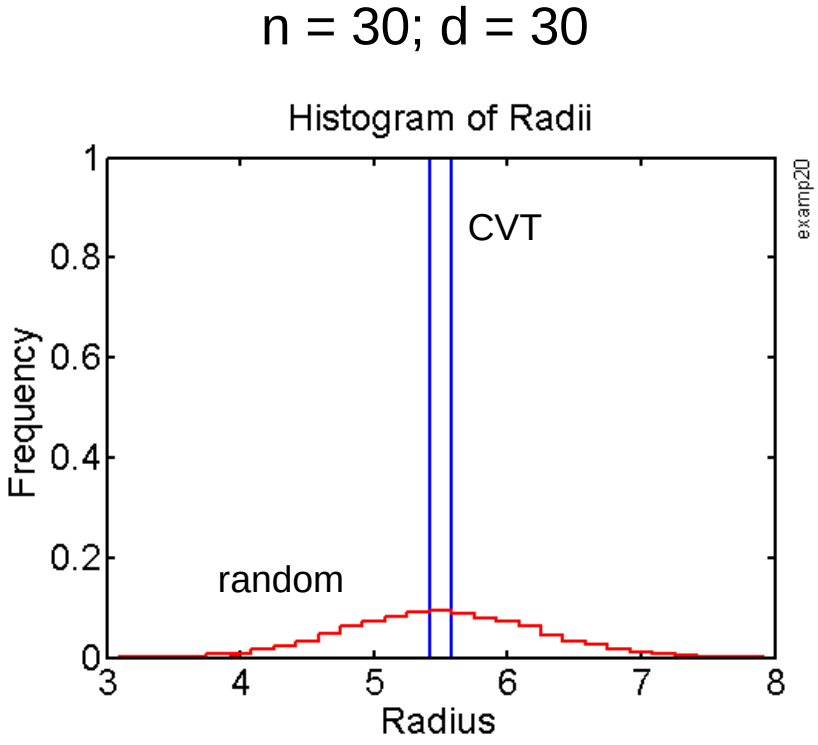
CVT, 30



All points are nearest neighbors!

CVT radial distribution: 30 points in 30D

- As with 2 points in 2D, all 30 CVT points in 30D are at the same radius
  - lie on the surface of a hypersphere
- As seen in last slide, the inter-point distances for CVT are essentially identical
  - regular point pattern (unique?)
- Rotation is only degree of freedom between different realizations of CVT
- One can generate new CVT patterns by randomly rotating an existing one

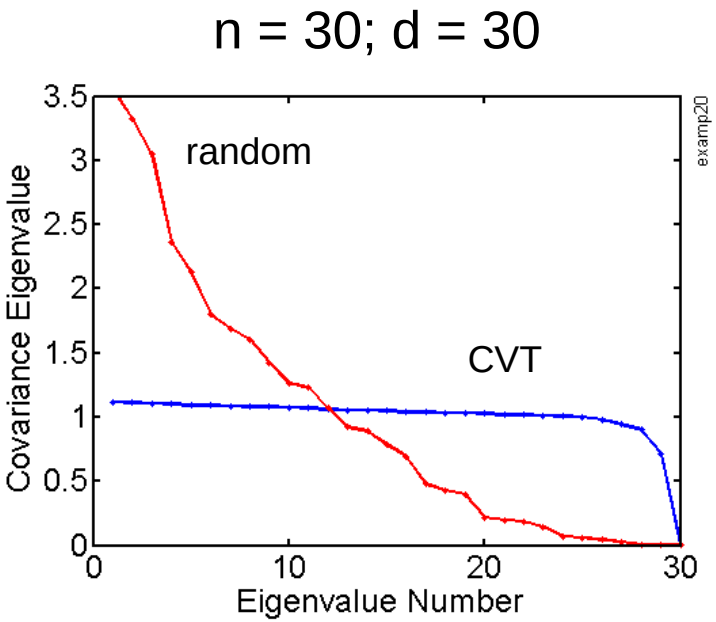


Covariance analysis of point set

- Let $\mathbf{x}^j$ be vector for jth point; point set is represented by matrix

$$\mathbf{X} = (\mathbf{x}^1; \mathbf{x}^2; \mathbf{x}^3; \cdots \mathbf{x}^n)^T$$

- Covariance of point set along the axes is $\mathbf{X}\mathbf{X}^T$
- Eigenanalysis of $\mathbf{X}\mathbf{X}^T$ yields the covariance spectrum
  - the ith eigenvalue is the variance of the points projected onto the ith eigenvector
- Conclude that spectrum for CVT point set is much more uniform than for random set, which is quite variable (the Wishart distribution)
- Last eigenvalue is zero; rank = 29



Linear response model

- Assume outputs of a simulation are linearly related to perturbations in the inputs, $\delta \boldsymbol{y} = \mathbf{S}_{\mathbf{y}}^{\mathrm{T}} \delta \mathbf{x}$
  where $\mathbf{S}_{\mathbf{y}}$ is sensitivity matrix $\partial \mathbf{y} / \partial \mathbf{x}$

- The covariance in the output **y** is

$$\mathbf{C}_{\mathbf{y}} = \mathbf{S}_{\mathbf{y}}^{\mathrm{T}} \mathbf{C}_{\mathbf{x}} \mathbf{S}_{\mathbf{y}}$$

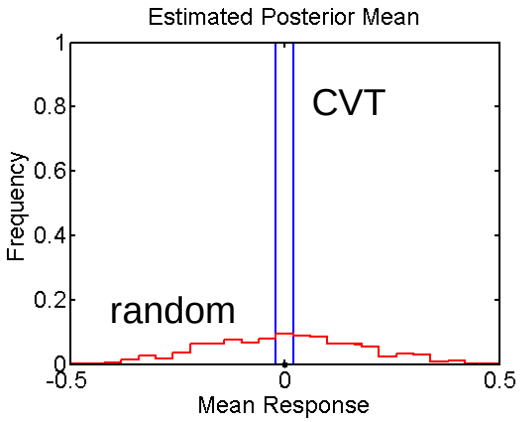
  - when output *y* is a scalar,
    the covariance $\mathbf{C}_{\mathbf{y}}$ is a scalar (variance),
    and $\mathbf{S}_{\mathbf{y}}$ is a vector

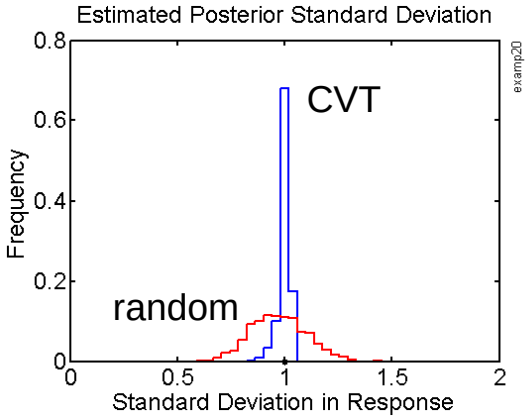
- If linear model is sufficient and one knows the sensitivity matrix, then predictive distribution is easily characterized
- However, for large simulations, the sensitivity matrix is often unknown

Test single point set using random sensitivities

- Assume linear model, $\delta \boldsymbol{y} = \mathbf{S}_{\mathbf{y}}^{\mathrm{T}} \delta \mathbf{x}$ where $\mathbf{S}_{\mathbf{y}}$ is sensitivity matrix $\partial \mathrm{y}/\partial \mathbf{x}$
- Test predictive response of a single sample set for an ensemble of random sensitivity vectors $\mathbf{S}_{\mathbf{y}}$:
  - $\mathbf{S}_{\mathbf{y}} = N_d(0,1)$; mean($\mathbf{S}_{\mathbf{y}}$) = 0 , var($\mathbf{S}_{\mathbf{y}}$) = 1
  - assume input **x** distribution is uncorrelated, unit-variance *d*–dimen. normal distribution, $N_d(0,1)$; $\mathbf{C}_{\mathbf{x}} = \mathbf{1}$
  - then expect: mean(**y**) = 0 , var(**y**) = 1
- Plots show CVT (blue) predicts mean and standard deviation of predictive distribution more accurately than random Monte Carlo (red)

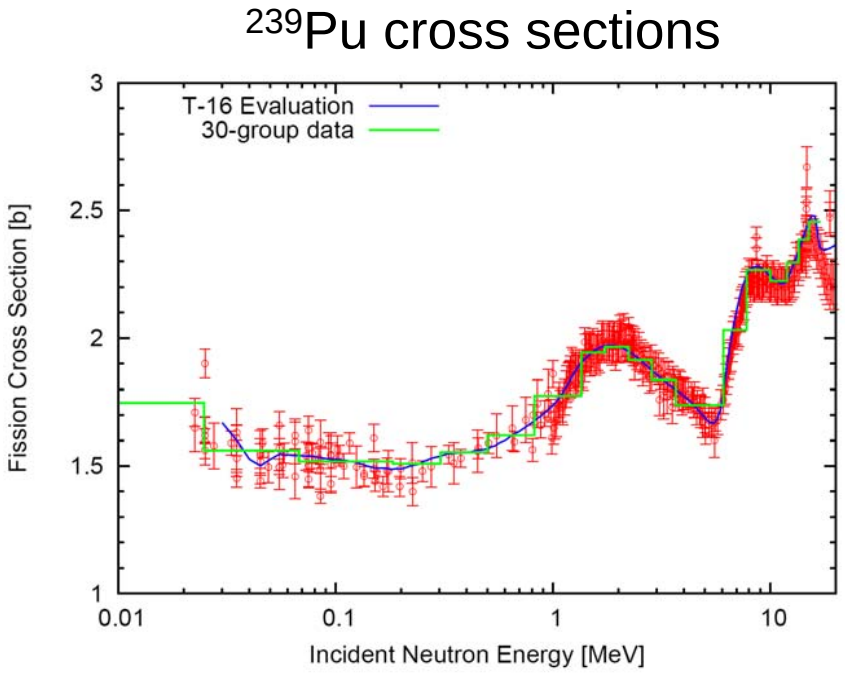
n = 30; d = 30





Neutron cross sections

- Plot shows
  - measured fission cross sections for neutrons on ^{239}Pu; red data points
  - inferred cross sections; blue line
  - weighted average in 30 energy bins (groups); green histogram
- PARITSN code simulates neutron transport based on multigroup, discrete-ordinates method
  - uses 30 energy bins
  - calculates criticality for specified configuration of fissile-material
  - establish dependence of criticality experiment to cross sections

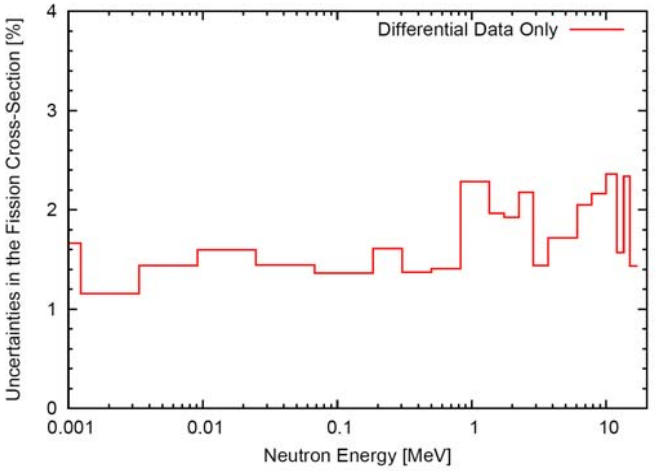


Neutron cross sections - uncertainties

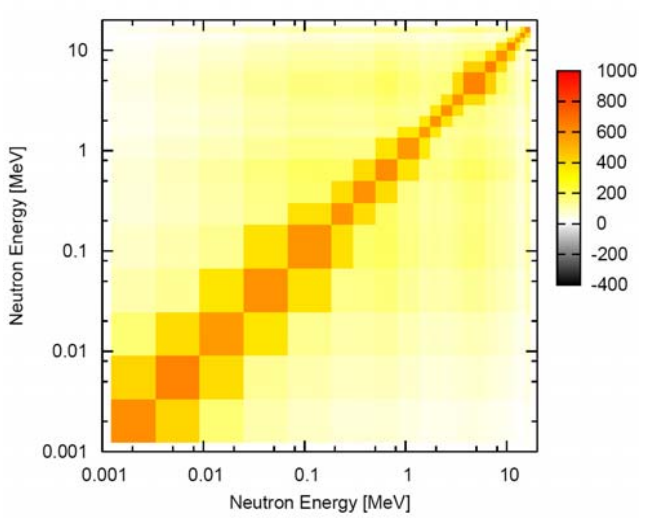
- Analysis of measured cross sections yields a set of evaluated cross sections

- Uncertainties in evaluated cross sections are ~ 1.4-2.4 %
- Covariance matrix important
- Strong positive correlations caused by normalization uncertainties in each experiment

standard error in cross sections



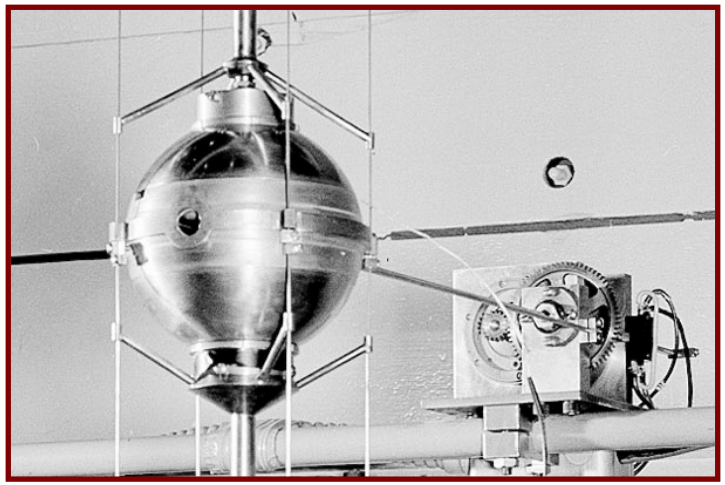
correlation matrix



JEZEBEL – criticality experiment

- JEZEBEL experiment (1950-60)
  - fissile material ^{239}Pu
  - measure neutron multiplication as function of separation of two hemispheres of material
  - summarize criticality with neutron multiplication factor, $k_{eff} = 0.9980 \pm 0.0019$
  - very accurate measurement

JEZEBEL set up

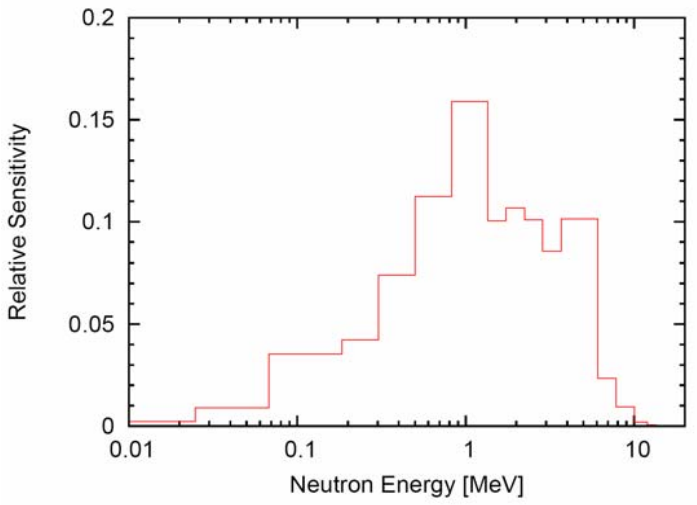


- Our goal – use highly accurate JEZEBEL measurement to improve our knowledge of ^{239}Pu cross sections

JEZEBEL – sensitivity analysis

- PARITSN code relates k_{eff} to neutron cross sections
- Sensitivity of k_{eff} to cross sections found by perturbing cross section in each energy bin by 1% and observing increase in k_{eff}
- Observe that 1% increase in all cross sections results in 1% increase in k_{eff} , as expected

k_{eff} sensitivity to cross sections



Bayesian update

- For data linearly related to the parameters, the Bayesian (aka Kalman) update is

$$\mathbf{C}_1^{-1}\mathbf{x}_1 = \mathbf{C}_0^{-1}\mathbf{x}_0 + \mathbf{S}_\mathbf{y}^{\mathrm{T}}\mathbf{C}_\mathbf{y}^{-1}\mathbf{S}_\mathbf{y}(\mathbf{y}-\mathbf{y}_0)$$

$$\mathbf{C}_1^{-1} = \mathbf{C}_0^{-1} + \mathbf{S}_\mathbf{y}^{\mathrm{T}}\mathbf{C}_\mathbf{y}^{-1}\mathbf{S}_\mathbf{y}$$

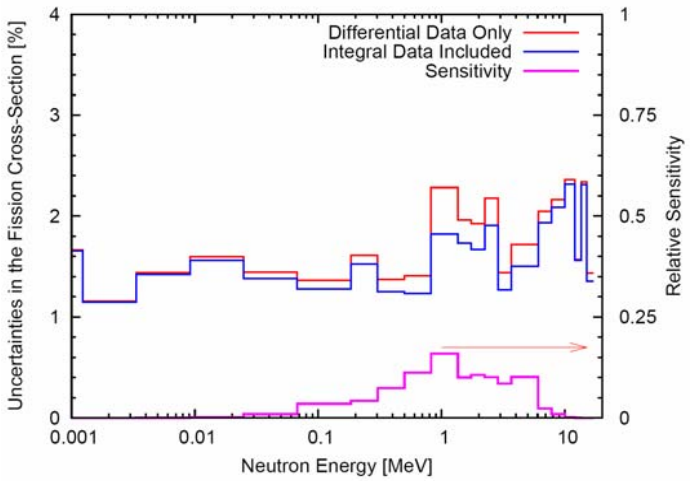
  - $\mathbf{x}_0$ and $\mathbf{x}_1$ are parameter vectors before and after update
  - $\mathbf{C}_0$ and $\mathbf{C}_1$ are their covariance matrices
  - $\mathbf{y}$ and $\mathbf{C}_\mathbf{y}$ are the measured data vector and its covariance
  - $\mathbf{y}_0$ is the value of $\mathbf{y}$ for $\mathbf{x}_0$
  - $\mathbf{S}_\mathbf{y}$ is the matrix of the sensitivity of $\mathbf{y}$ to $\mathbf{x}$; $\partial\mathbf{y}/\partial\mathbf{x}$

- For the JEZEBEL case, $\mathbf{y}$ is a scalar (k_{eff}), $\mathbf{C}_\mathbf{y}$ is a scalar (variance), and $\mathbf{S}_\mathbf{y}$ is a vector

Updated cross sections

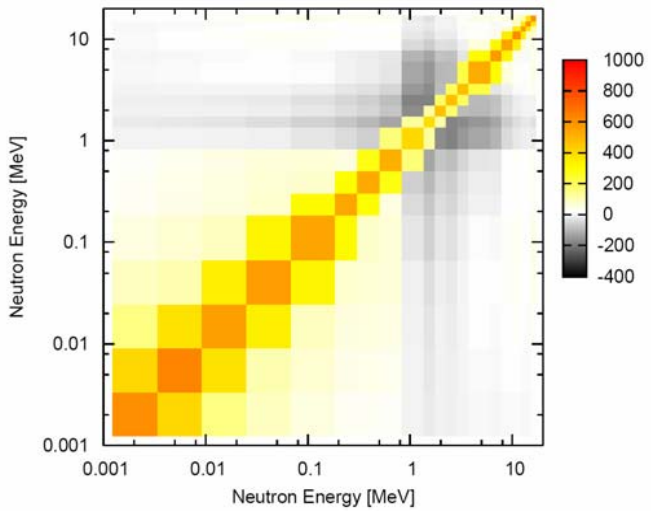
- Plot shows uncertainties in cross sections before and after using JEZEBEL measurement
- Modest reduction in uncertainties; follows energy dependence of sensitivity

standard error in cross sections



- Correlation matrix is significantly altered
- Strong negative correlations introduced by integral constraint of matching JEZEBEL's k_{eff}

correlation matrix

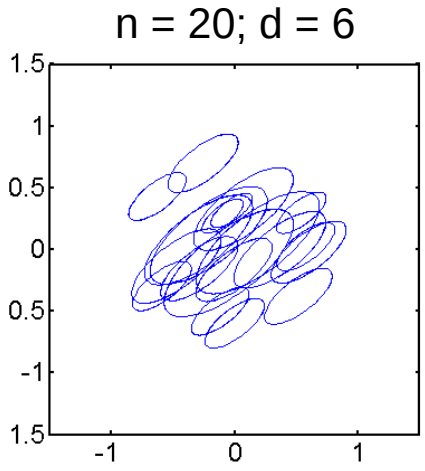


Uncertainty in subsequent simulations

- Intend to use updated cross sections in new calculations, with expectation that integral constraint will reduce uncertainties
- Next, demonstrate use of Monte Carlo sampling for estimating the uncertainty in new predictions; use random MC or qMC?
- Demonstrate usefulness of quasi-MC in form of CVT point sets by "predicting" k_{eff} measured in JEZEBEL
  - for this demo, assume linear model with known sensitivity vector
  - under these assumptions, we can calculate exact answer and compare to MC-style sampling to obtain predictive distribution
- For new physical scenario, we would not have sensitivity vector and we would have to do full simulation calculation, so only a modest number of function evaluations can be done

Rotation matrix in high dimensions

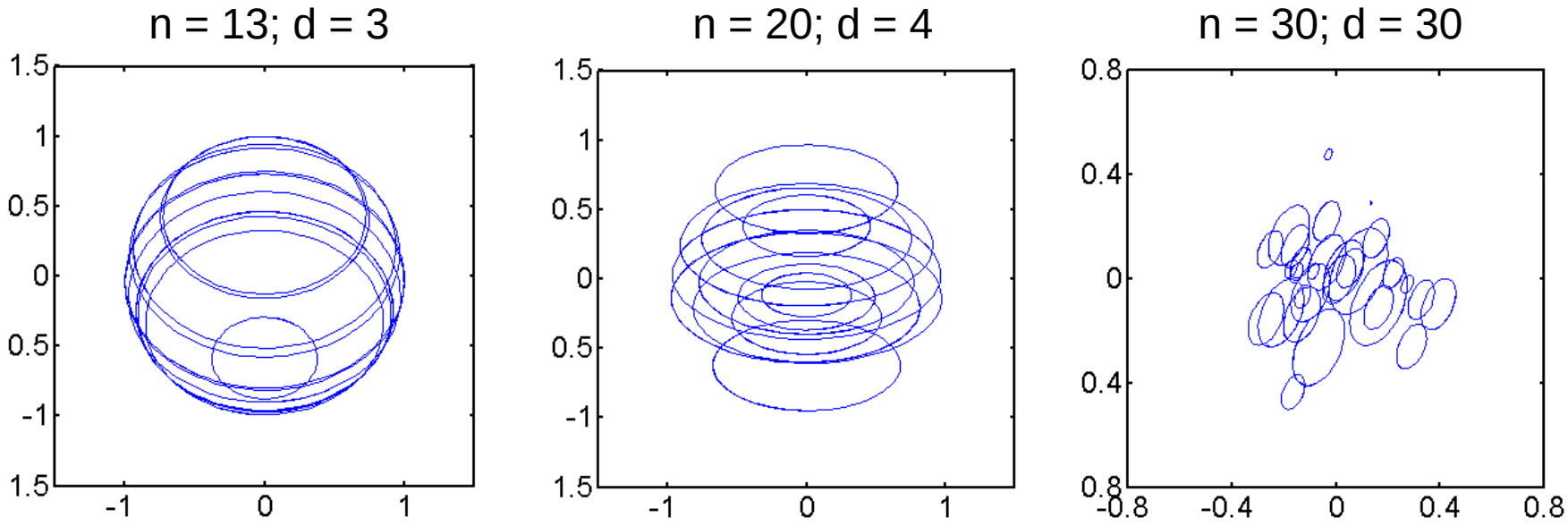
- Given unit vectors **a** and **b**, want rotation matrix that map **a** into **b**
- Algorithm (thanks to Mike Fitzpatrick)
  - for matrix $\mathbf{M}_1 = (\mathbf{a}; \mathbf{b})^{\mathrm{T}}$
  - Singular Value Decomposition (SVD) of:
    $$\mathbf{M}_1 = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$$
  - the bases of the subspace orthogonal to **a** and **b** are given by singular vectors in **U**, except for first two
  - then, do SVD on outer product matrix $\mathbf{M}_2 = \mathbf{B}\mathbf{A}^{\mathrm{T}}$
    $$\mathbf{A} = (\mathbf{a}; \mathbf{u}^3; \mathbf{u}^4; \cdots \mathbf{u}^{\mathrm{d}})^{\mathrm{T}} \; ; \quad \mathbf{B} = (\mathbf{b}; \mathbf{u}^3; \mathbf{u}^4; \cdots \mathbf{u}^{\mathrm{d}})^{\mathrm{T}}$$
  - rotation matrix is $\mathbf{R} = \mathbf{U}_2 \mathbf{D} \mathbf{V}_2^{\mathrm{T}}$ where **D** is identity matrix, except $[\mathbf{D}]_{d\,d} = \det(\mathbf{U}_2 \mathbf{V}_2^{\mathrm{T}})$



- Random rotations – randomly choose directions of **a** and **b**
  - simple algorithm: randomly draw vector from normal distribution and normalize it to unit length

Examples of rotations in high dimensions

- Random rotations is various dimensions
- All points have unit radius (on surface of unit hypersphere)



Sampling from correlated normal distribution

- Want to draw samples from multi-variate normal distribution with known covariance $\mathbf{C}_\mathbf{x}$
- Important to include correlations among uncertainties, i.e., off-diagonal elements
- Algorithm:
  - perform eigenanalysis of covariance matrix of *d* dimensions

$$\mathbf{C}_\mathbf{x} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\mathrm{T}$$

    where **U** is orthognal matrix of eigenvectors and
    $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues
  - draw *d* samples from unit variance normal distribution, ξ_i
  - scale this vector by $\lambda_i^{1/2}$
  - transform vector into parameter space using the eigenvector matrix
  - to summarize: $\mathbf{x} = \mathbf{U}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\xi}$

Accuracy of predicted k_{eff} and its uncertainty

- Check accuracy of predicted mean and standard deviation based on 30 samples; CVT vs. random sample sets
  - exact value from known sensitivity and linear model
- Conclude – CVT is more accurate here than random sampling

Results from 1000 sample sets; 'rot' indicates single sample set randomly rotated to achieve each new one

	est. mean k_{eff}		est. std. dev. k_{eff}	
	avg.	rms dev.	avg.	rms dev.
random	0.99788	0.00037	0.00191	0.00028
random-rot	0.99824	0.00010	0.00218	0.00010
CVT-rot	0.99796	0.00001	0.00197	0.00002
exact-linear	0.99796	-	0.00195	-

Summary

- CVT sampling is useful in predictive sampling for obtaining higher accuracy for a limited number of simulations
- CVT and repulsive particle model may be used to generate QMC point sets
  - particularly useful for modest number of points

Future work

- Need a way to estimate accuracy of results, a problem common to all QMC approaches
- Important to learn to cope with under-sampling ($n < d + 1$)
  - a parameter subspace dependence is not sampled so some aspects of uncertainty in input parameters may be missed
  - satisfactory solution requires careful thought, taking into account what is known about the influence of input parameters on the simulation output
- Sequential generation of point sets
  - add additional points, keeping previous points fixed
- Adaptive sampling – improve estimate by importance sampling, i.e., increasing density of points in selected regions
- Employ advanced analysis of outputs produced by input samples
  - weighted Monte Carlo
  - characterize output response as function of inputs

Future work

- Semi-random-quasi-Monte Carlo
  - partially randomize quasi-Monte Carlo
  - strike balance between random and deterministic patterns
  - try to achieve the benefits of each approach

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