Algebraic Singularity Method for Mass Measurements with Missing Energy

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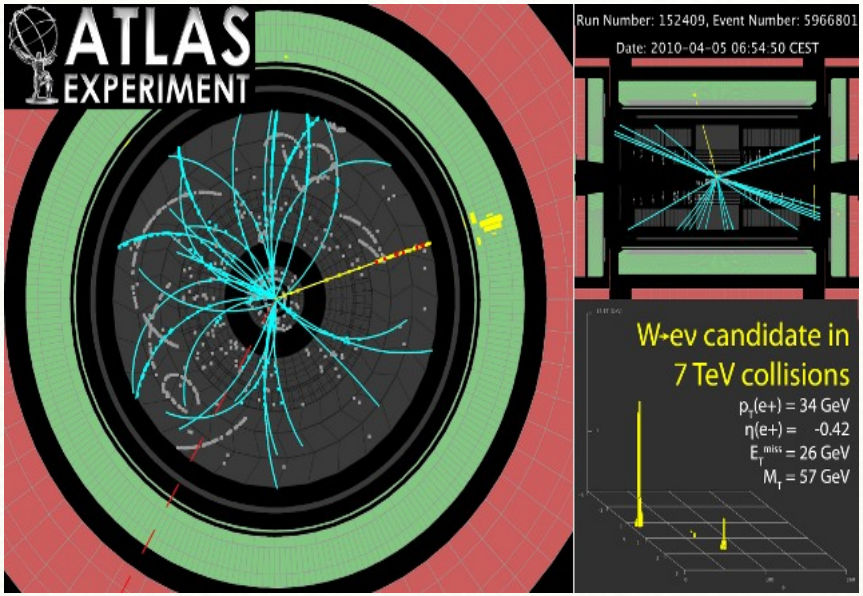
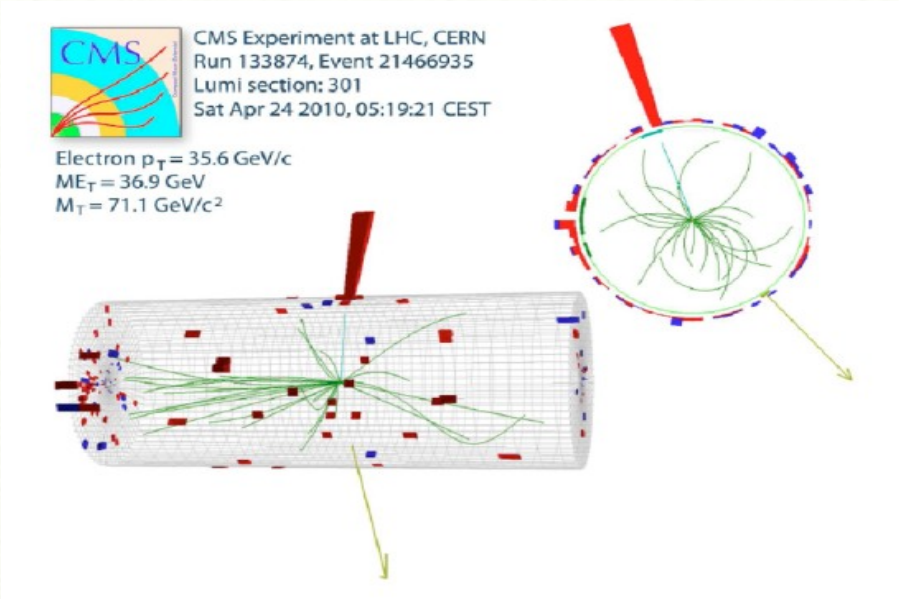
PRL 104, 081601 (2010) (arXiv:0910.1147)
and work in progress

Santa Fe 2010 Jul 8, 2010

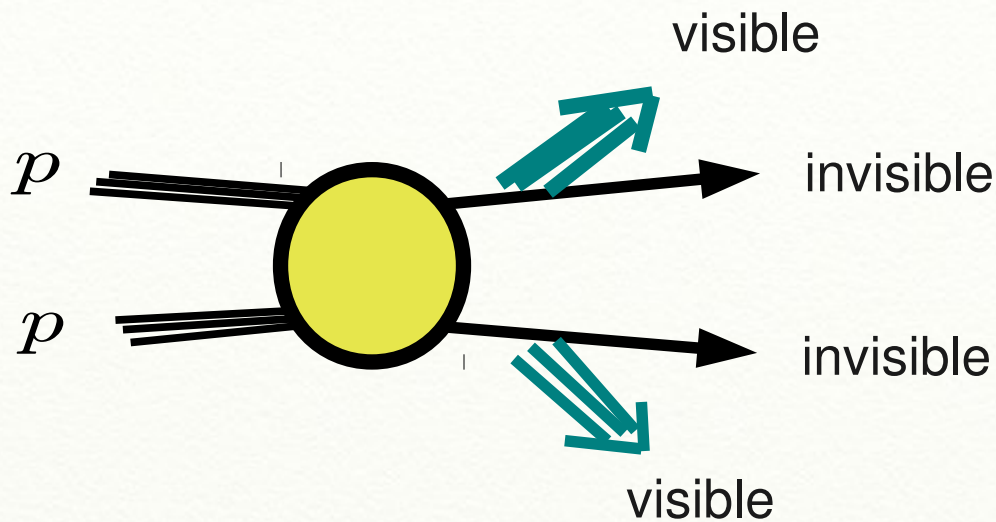
Introduction

First W candidates in ATLAS and CMS at 7 TeV





- Prepare for discovery of Dark Matter particle!
- Search for missing energy signatures
  (“invisible” or “missing” particles)



Missing particles
are often pair-produced.

↓

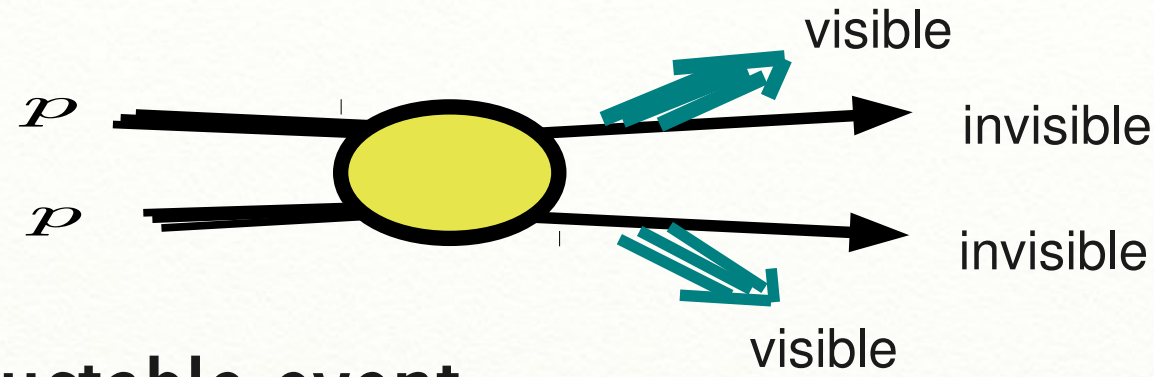
Mass Measurements are Challenging.

Many new physics model have new symmetries:

- prohibits dangerous decays
- consistent with electroweak precision data

→ stable neutral particles (DM candidates!)

Prototype examples : MSSM with R-parity, LH with T-parity, UED with KK-parity



- **Reconstructable event** :

  Nojiri, Polesello, Tovey (2003)
  Cheng, Gunion, Han, Marandella, McElrath (2007)
  Cheng, Engelhardt, Gunion, Han, McElrath (2008)

- **Nonreconstructable event** :

  - Use end point or cusps of kinematic variable

    Hinchliffe, Paige, Shapiro, Soderqvist, Yao (1997)
    Han, IWK, Song (2009)

  - Transverse Mass mT2 : end point → kink

    Lester, Summers (1999), Cho, Choi, Kim, Park (2007), Barr, Gripaios, Lester (2007)
    Tovey (2008), Cheng, Han (2008), Konar, Kong, Matchev, Park (2009)
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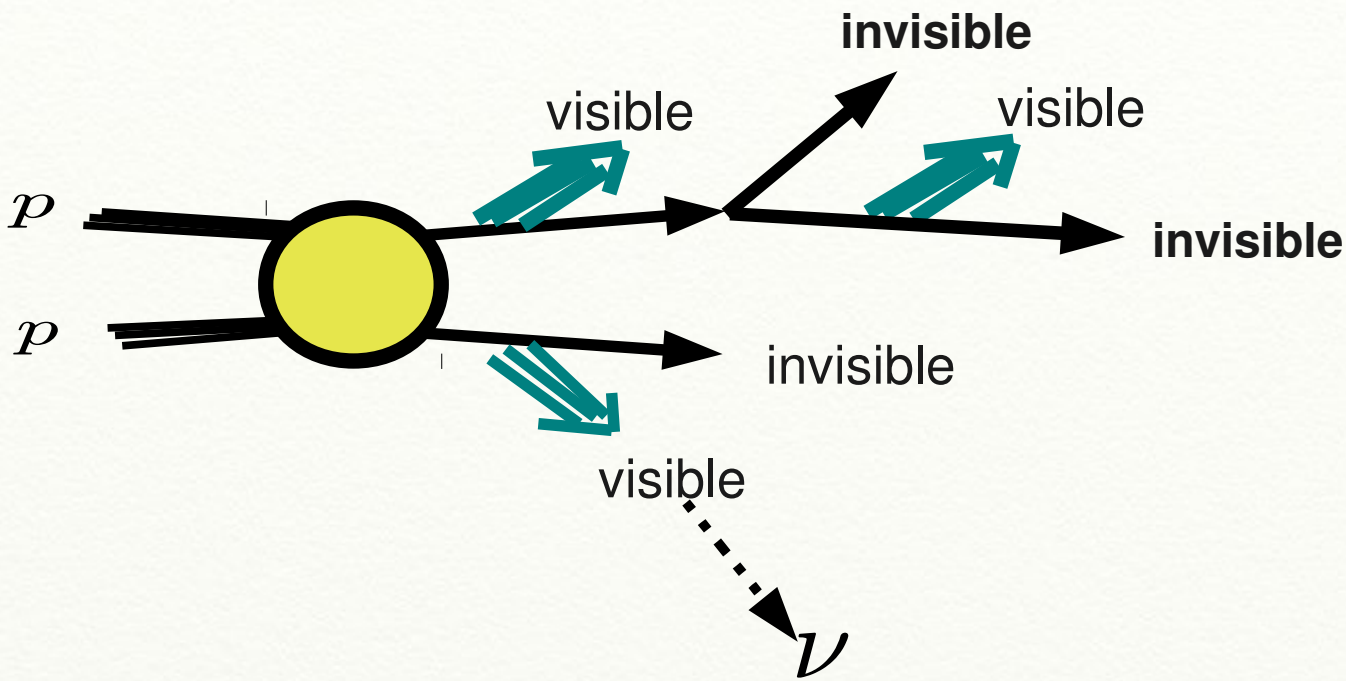
See the review by Barr and Lester: arXiv:1004.2732

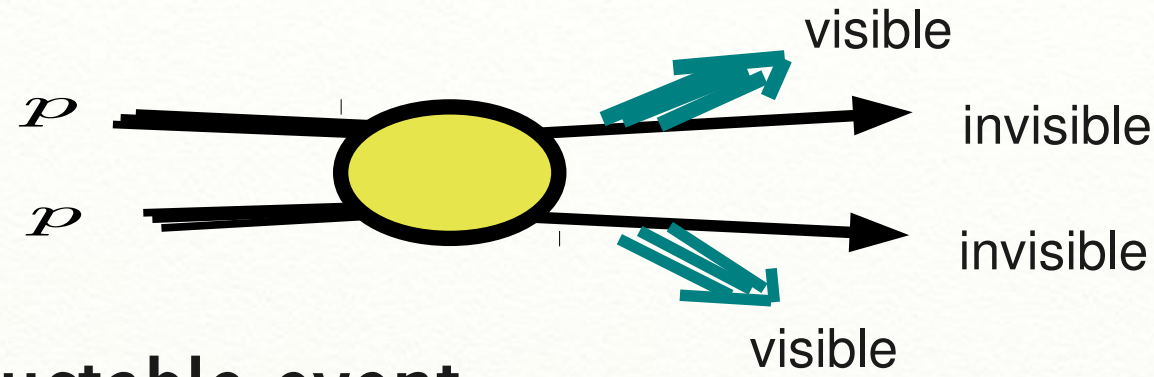
Event topology may be even more complex.

(more missing particle including neutrino, asymmetric chain)



Do we have a generalized mass measurement method?

<u>What is the essence of missing mass measurements?</u>



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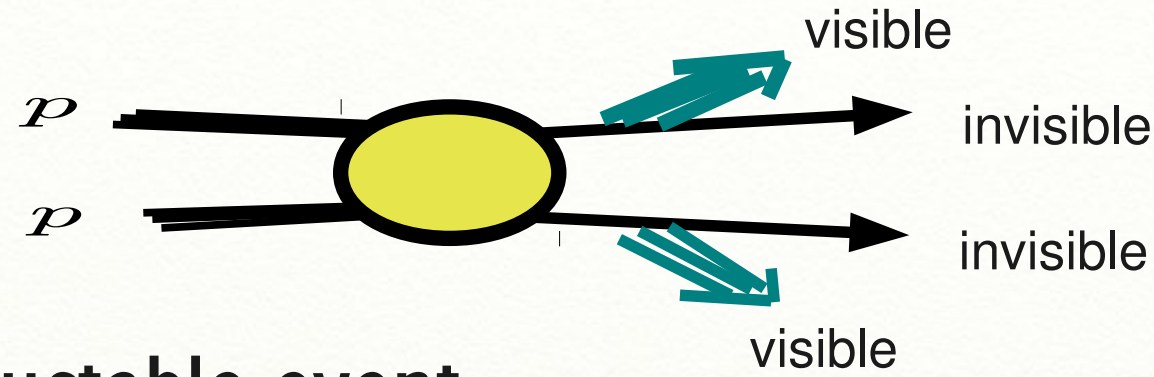

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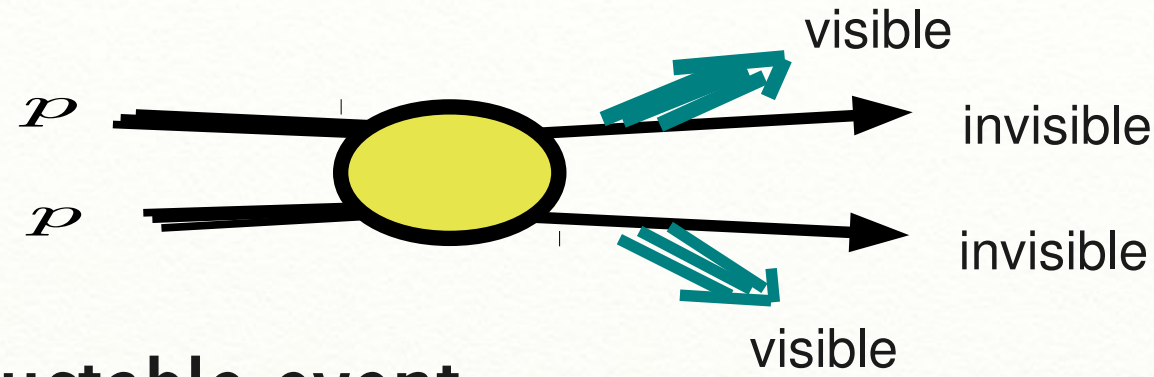

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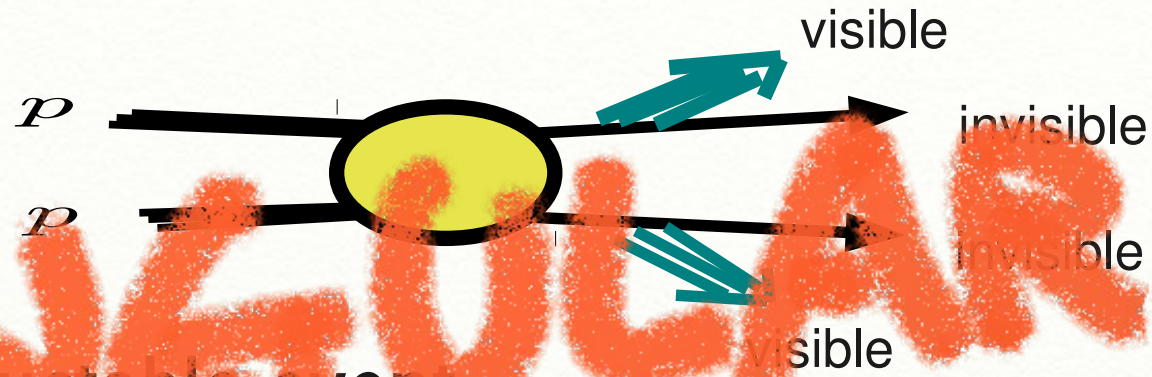

SINGULARITY

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NONSMOOTH

- **Non reconstructable event :**

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POINT

  - Transverse Mass mT2 : end point → kink

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IN DISTRIBUTION

Algebraic Singularity Method

- Approaches Mass Measurements in a disciplined way

- Based on the observation that mass measurements rely on singularity structures in PS

- Systematic and mathematical formulation and generalized approach

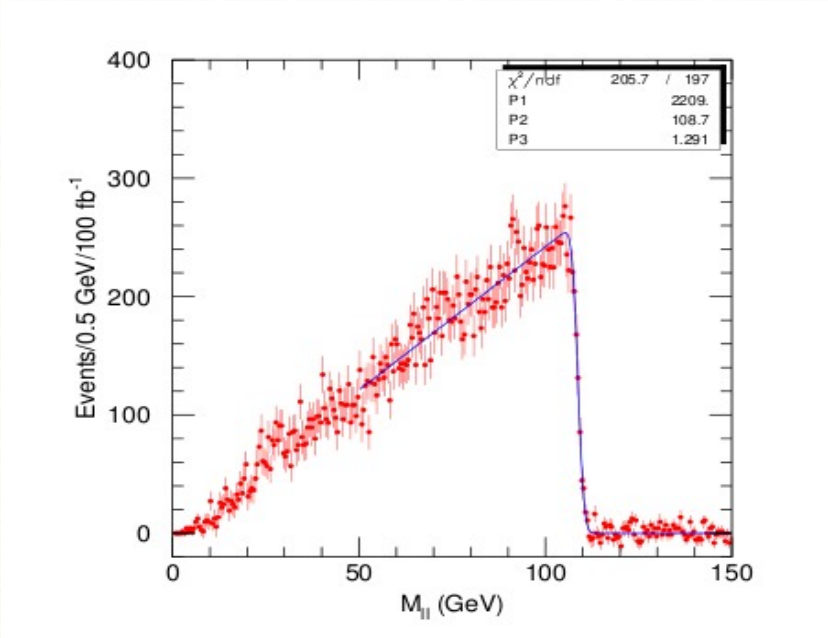
- Provide an optimized kinematic variable:
  "singularity coordinate"

- Subsumes the previous known techniques and gives a better understanding.

I will show ...

- Singularities in Kinematic Distributions
- Algebraic Singularity Method
- Ex) Reanalysis of cascade decay
- Singularity Coordinate
- Ex) Double Missing Particle Chain Topology

Ex1) End point singularity in invariant mass distribution

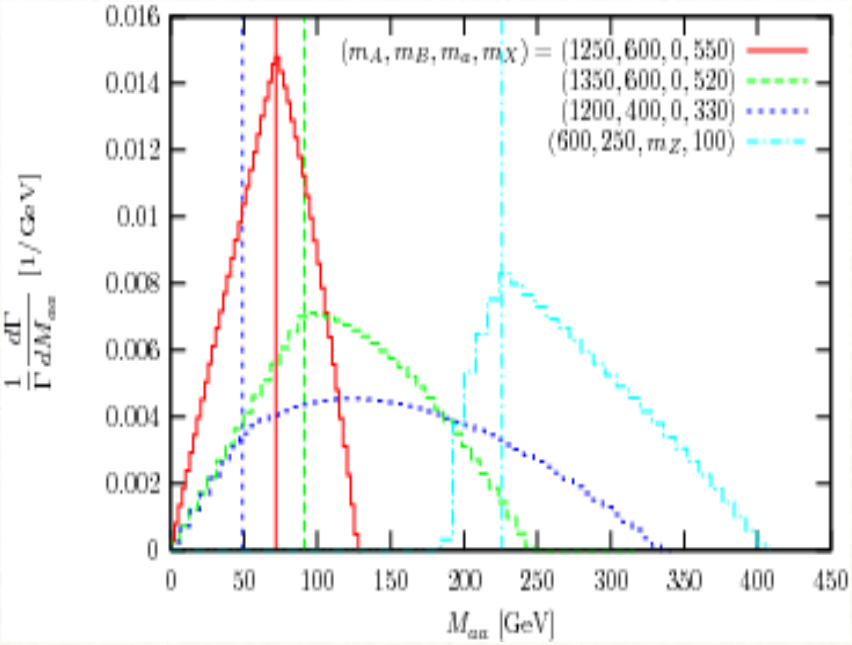


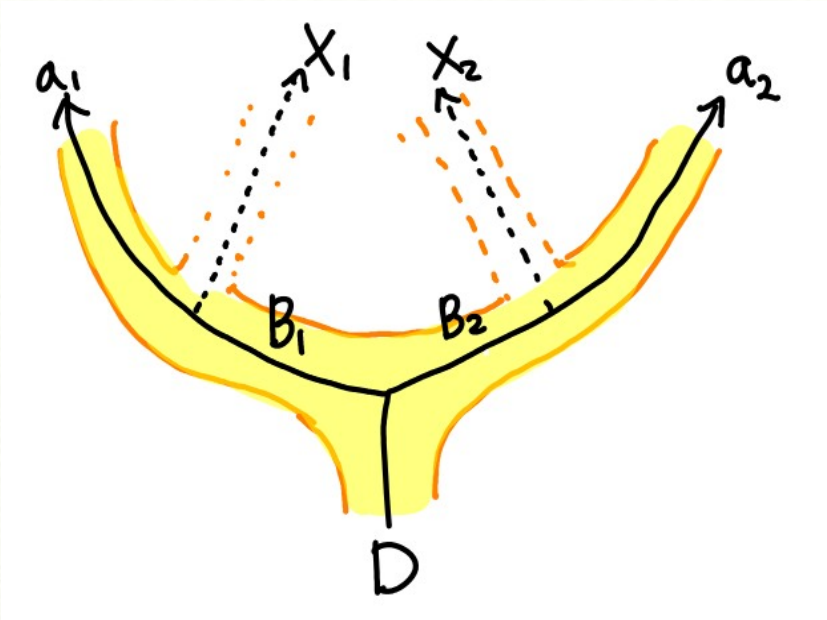
$$m_{ll} = \sqrt{(l_n + l_f)^\mu (l_n + l_f)_\mu}$$

from Bachacou, Hinchliffe, Paige (1999)

$$m_{ll}^{\text{edge}} = \sqrt{\frac{(m_Z^2 - m_Y^2)(m_Y^2 - m_X^2)}{m_Y^2}}$$

Ex2) Cusp and End point singularity in inv mass dist.





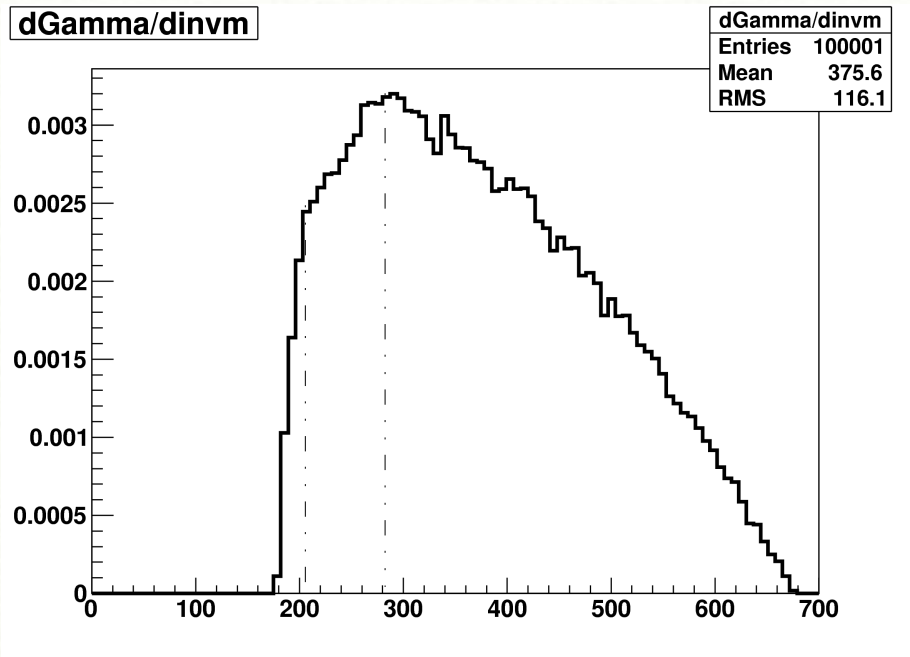
Han, IWK, Song (2009)

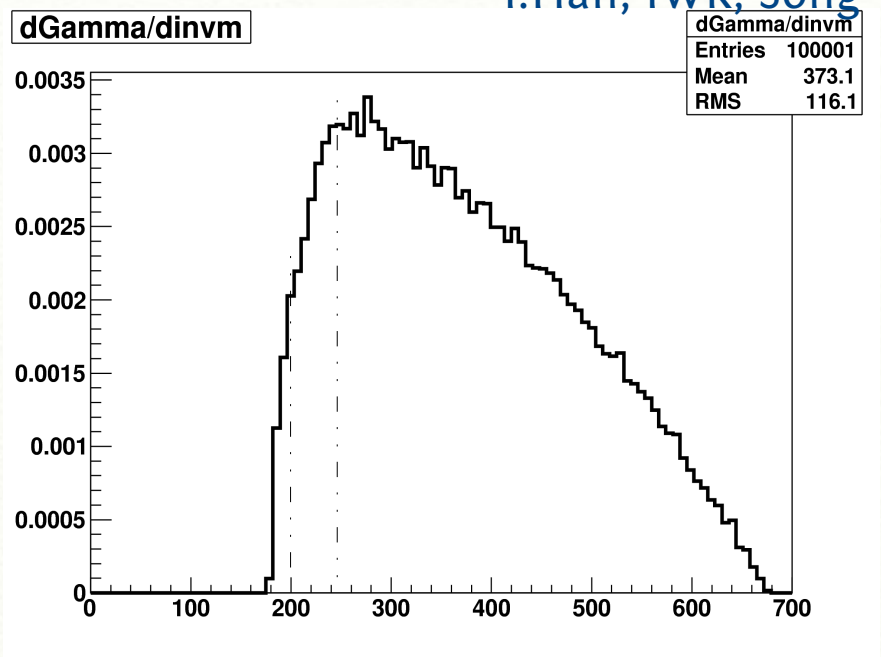
$$\frac{M_{aa}^{\rm cusp}}{M_{aa}^{\rm max}} = \exp(-2\eta) = \frac{m_D^2 - 2m_B^2}{2m_B^2} - \frac{m_D}{m_B}\sqrt{\frac{m_D^2}{4m_B^2} - 1}$$

$$M_{aa}^{\rm cusp} M_{aa}^{\rm max} = m_B^2 \left(1 - \frac{m_X^2}{m_B^2}\right)^2$$

More non-smooth structures w/ massive visible particles

T.Han, IWK, Song





$$\chi \equiv \frac{M_{aa}^2}{(2m_a^2)} - 1$$

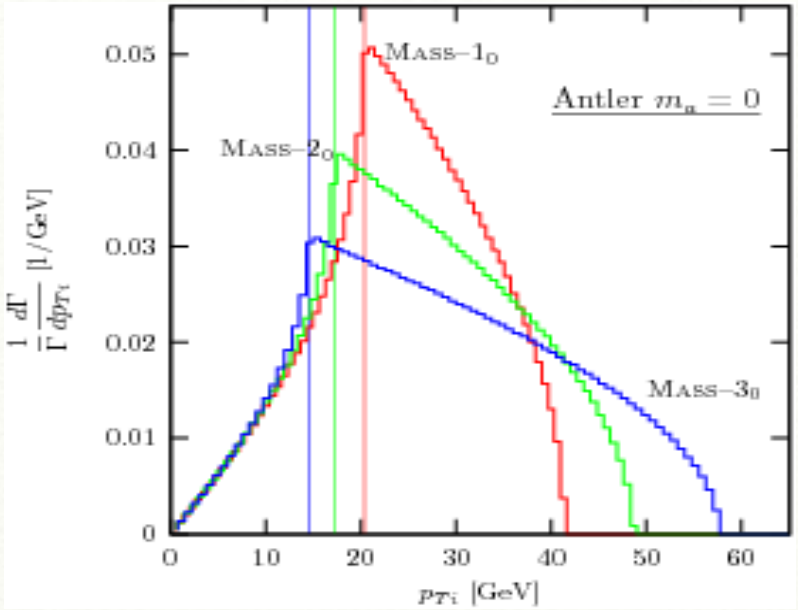
$$\left.\frac{d\Gamma}{d\chi}\right|_{\eta \le \frac{\zeta}{2}} \propto \begin{cases} 2\cosh^{-1}\chi, & \text{if } 1 \le \chi \le c_\eta; \\ 4\eta, & \text{if } c_\eta \le \chi \le c_-; \\ 2\zeta + 2\eta - \cosh^{-1}\chi, & \text{if } c_- \le \chi \le c_+, \end{cases}$$

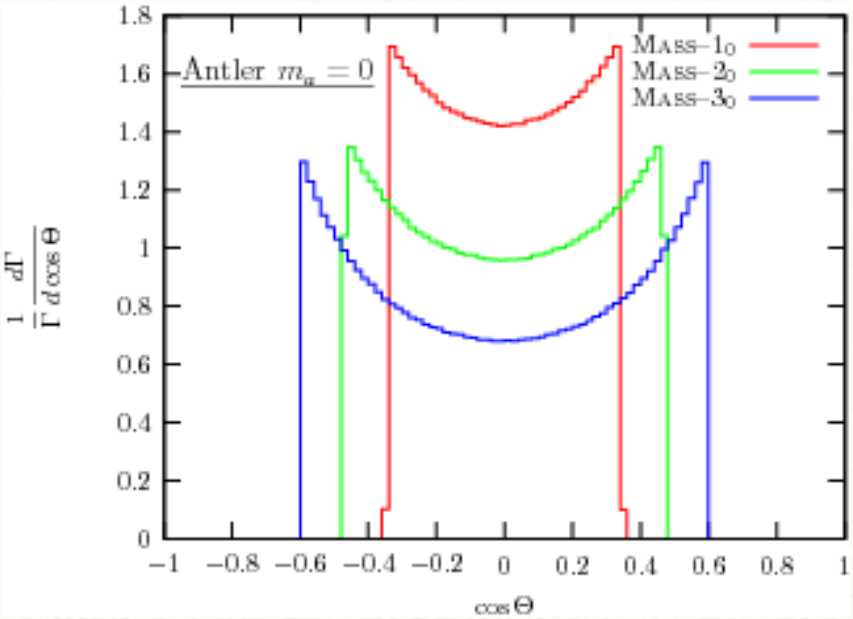
$$\left.\frac{d\Gamma}{d\chi}\right|_{\frac{\zeta}{2} < \eta < \zeta} \propto \begin{cases} 2\cosh^{-1}\chi, & \text{if } 1 \le \chi \le c_-; \\ 2\zeta - 2\eta + \cosh^{-1}\chi, & \text{if } c_- \le \chi \le c_\eta; \\ 2\zeta + 2\eta - \cosh^{-1}\chi, & \text{if } c_\eta \le \chi \le c_+, \end{cases}$$

$$c_+ = \cosh 2(\eta + \zeta) \qquad c_- = \cosh 2(\eta - \zeta) \qquad c_\eta = \cosh 2\eta$$

Such singularities also appear in other distributions.

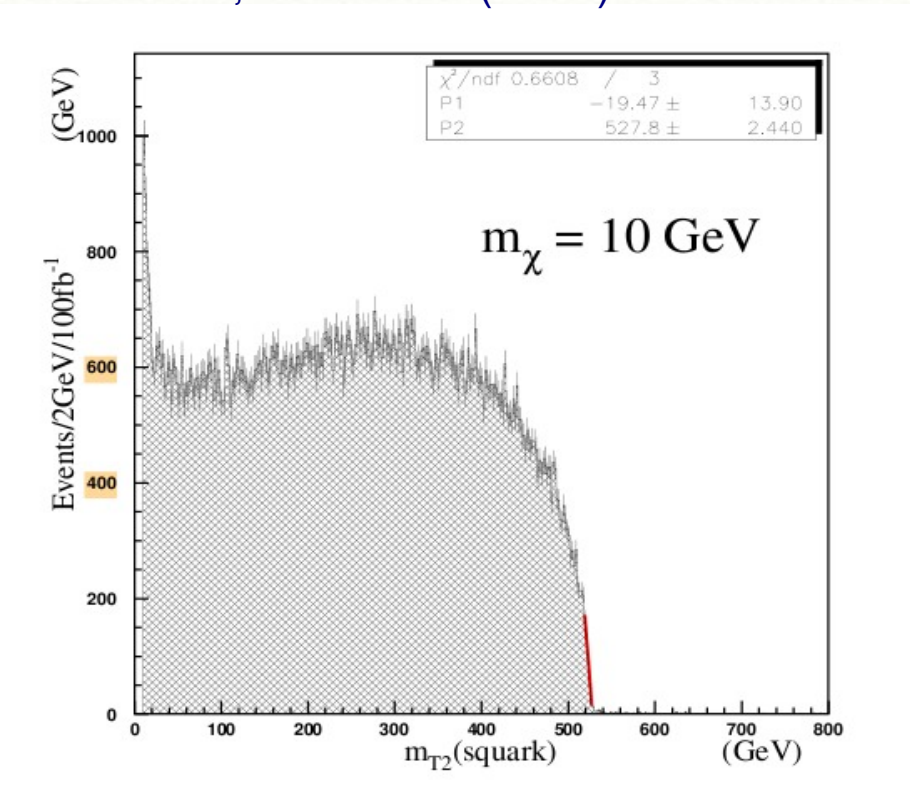
(T. Han, IWK, Song, work in progress)





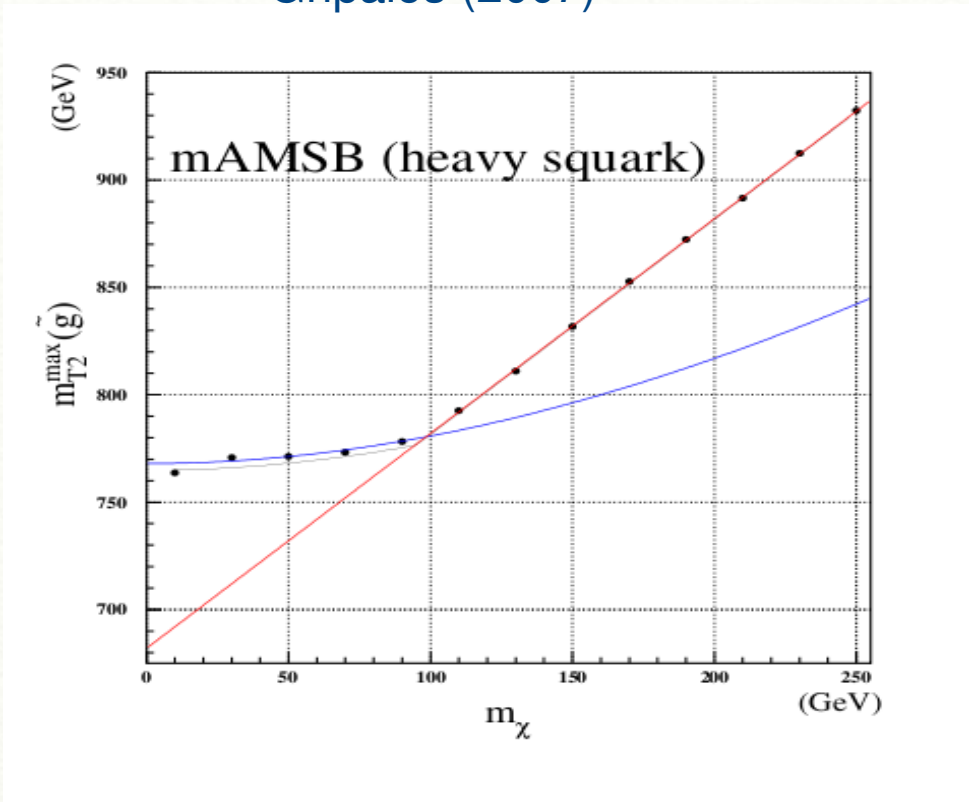
Ex3) m_{T2} end point and a kink in $m_{T2}^{(\max)}$

Lester, Summers (1999)



Cho, Choi, Kim, Park (2007)

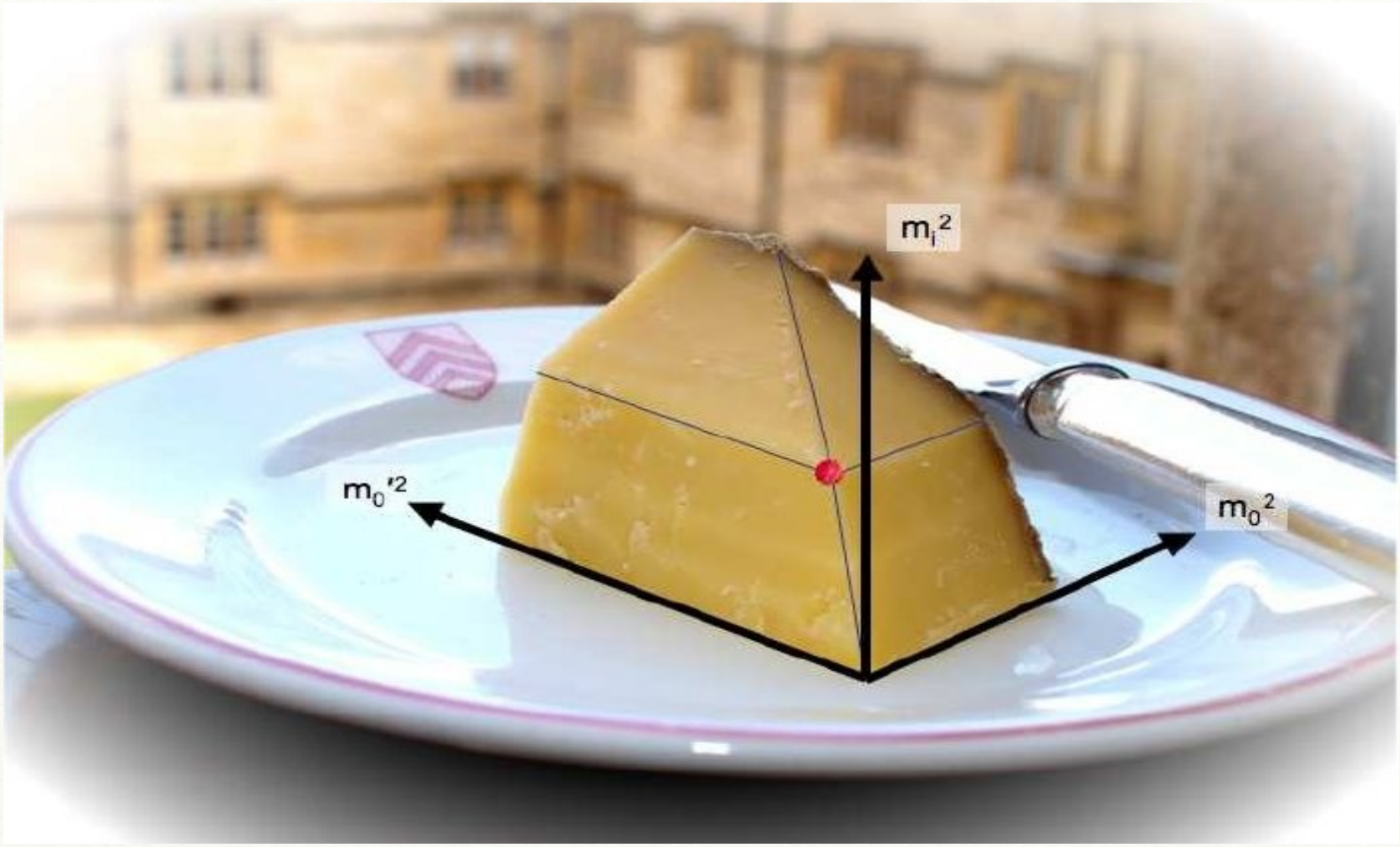
Gripaios (2007)



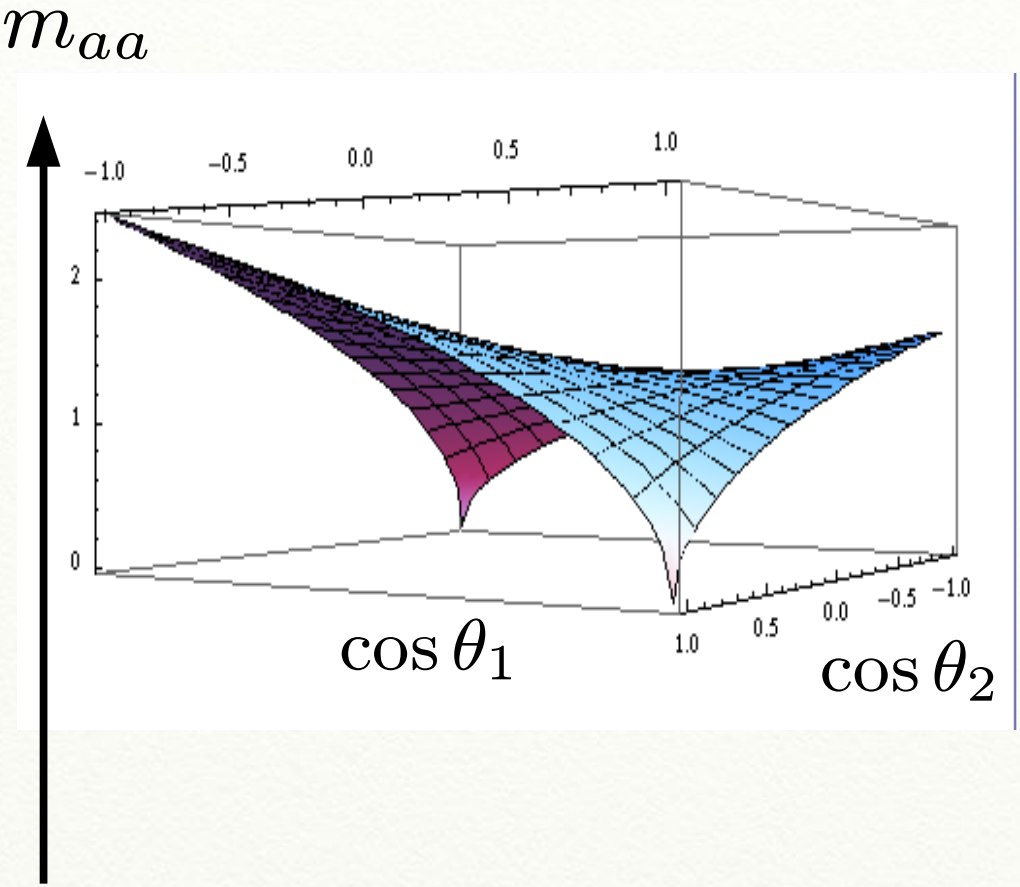
Figures from Cho, Choi, Kim, Park (2007)

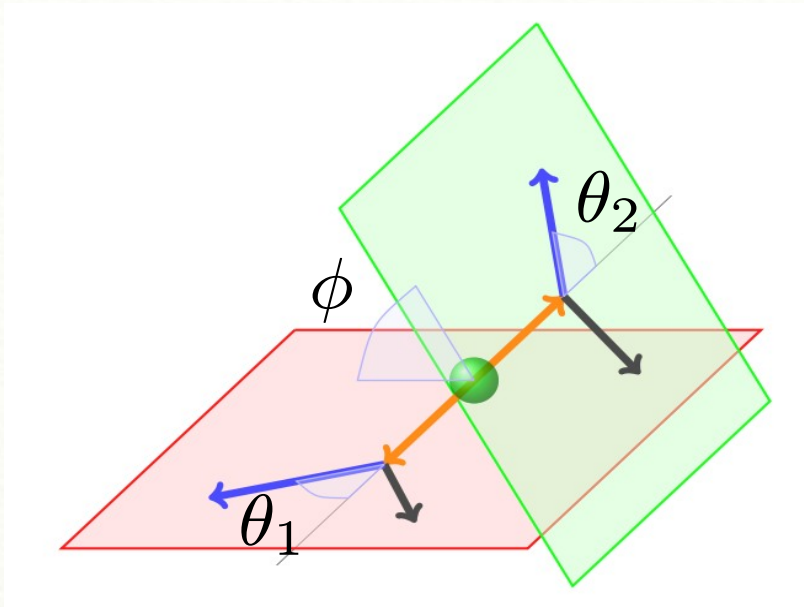
m_{T2} boundary / crease

Barr, Gripaios, Lester (2009)

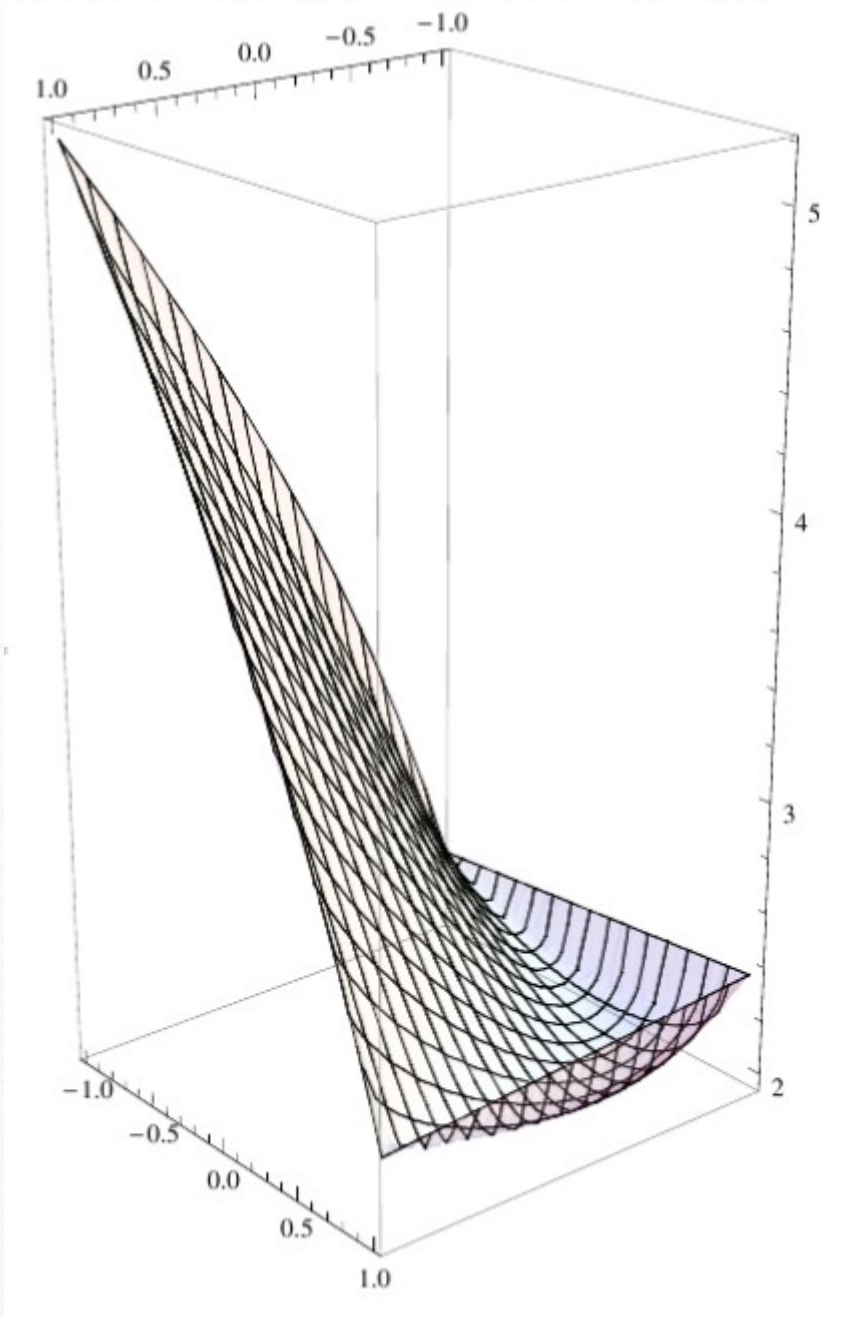


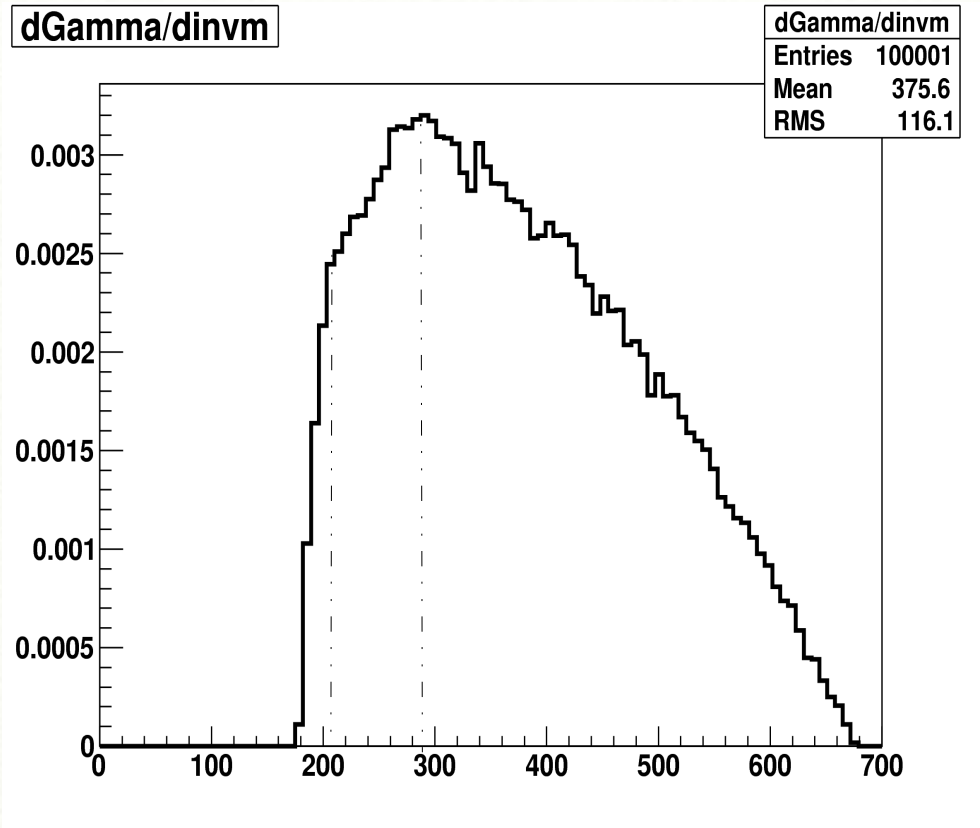
Why do the singularities appear?



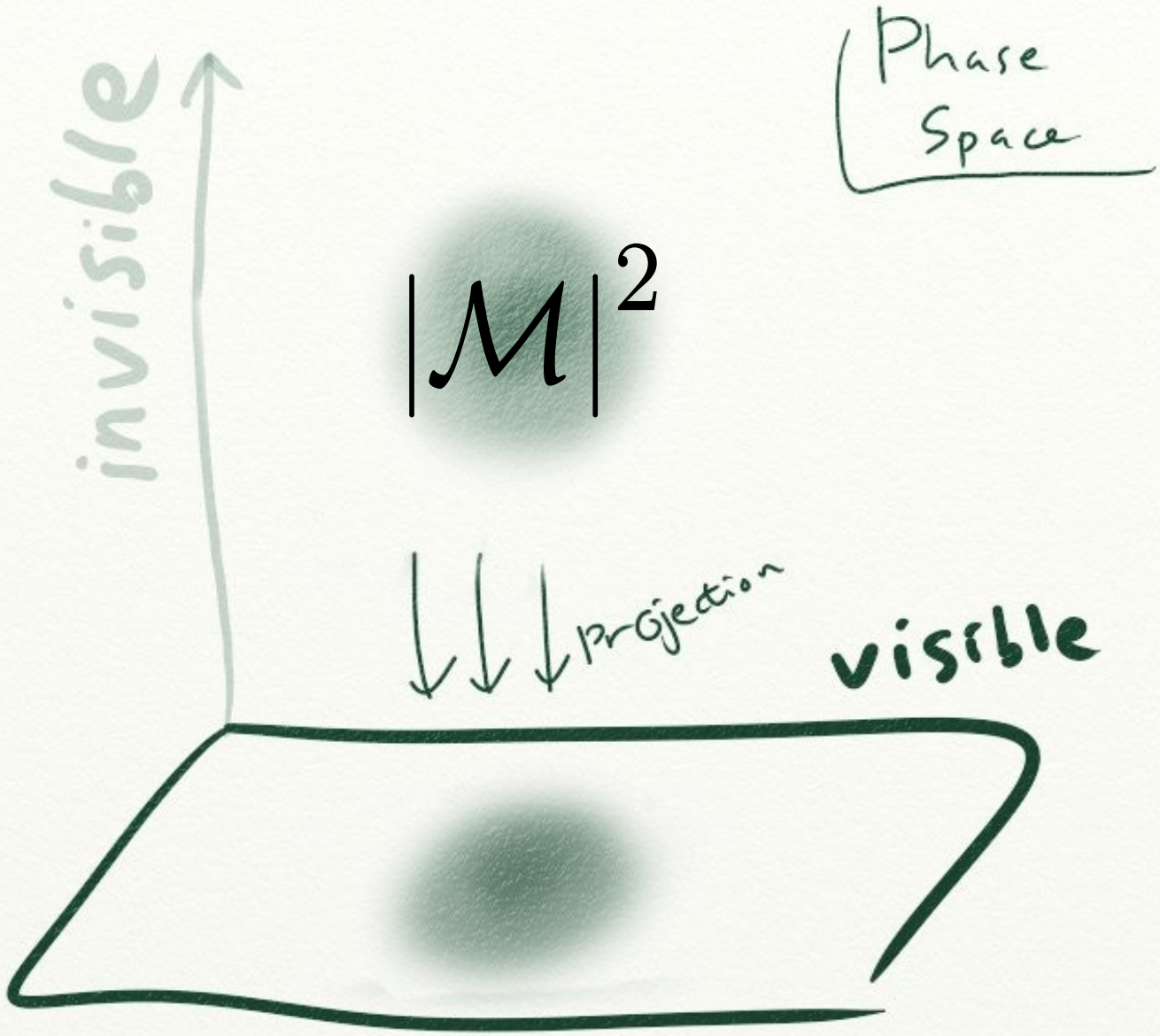


Phase Space is folded for a kinematic variable.

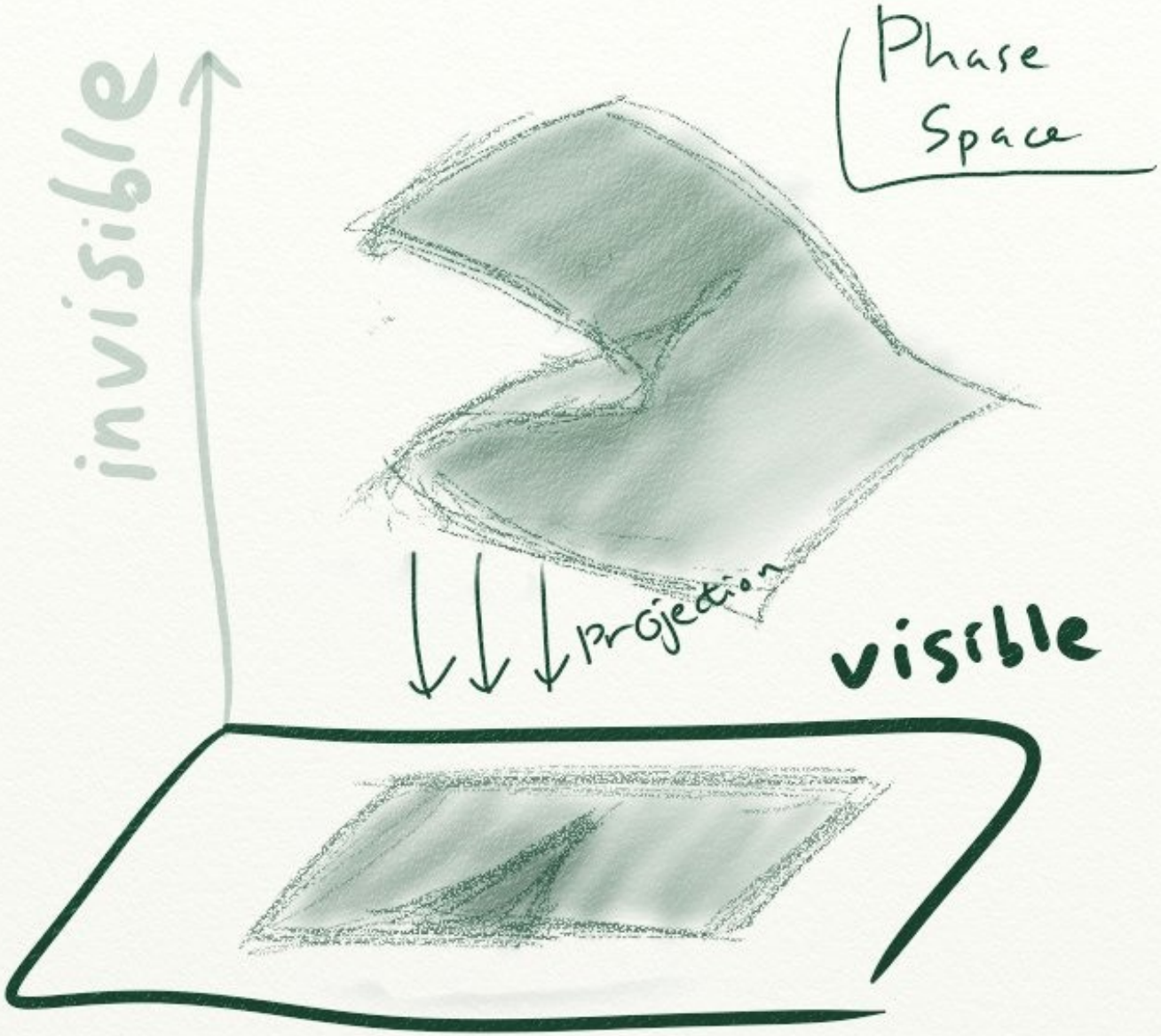

1.0
0.5
0.0
−0.5
−1.0
5
4
3
2
−1.0
−0.5
0.0
0.5
1.0


dGamma/dinvm
dGamma/dinvm
Entries 100001
Mean 375.6
RMS 116.1
0.003
0.0025
0.002
0.0015
0.001
0.0005
0
0
100
200
300
400
500
600
700

Singularities in Kinematic Distributions

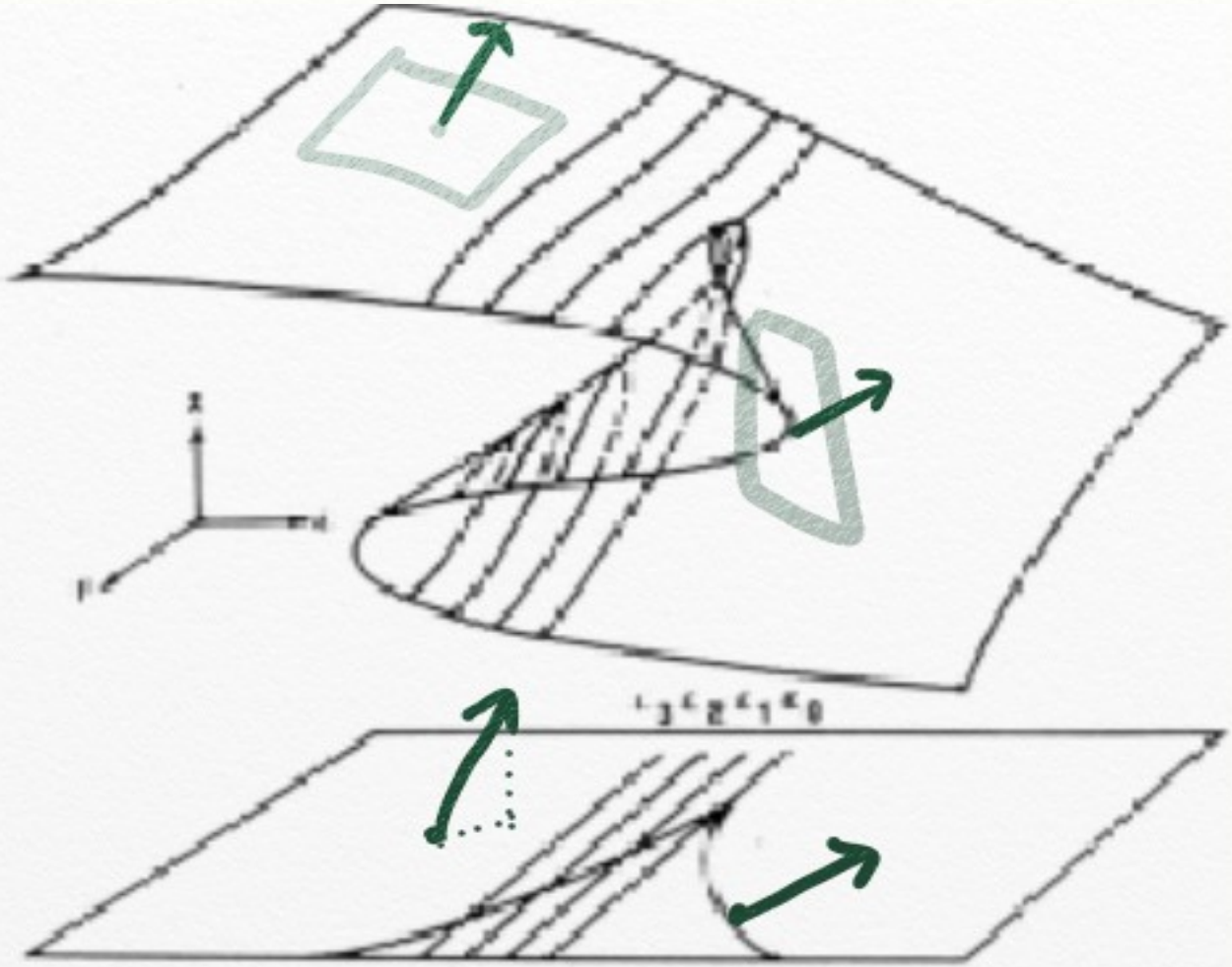


Phase Space is condensed to restricted submanifold in presence of on-shell particles



Algebraic Singularity Method

IWK (2009)



Singularities happen when the tangent space is aligned along invisible momentum direction.

$$\int d\mathrm{PS} = \int \ldots \int d^4p \ldots \delta(g_1)\delta(g_2)\ldots$$

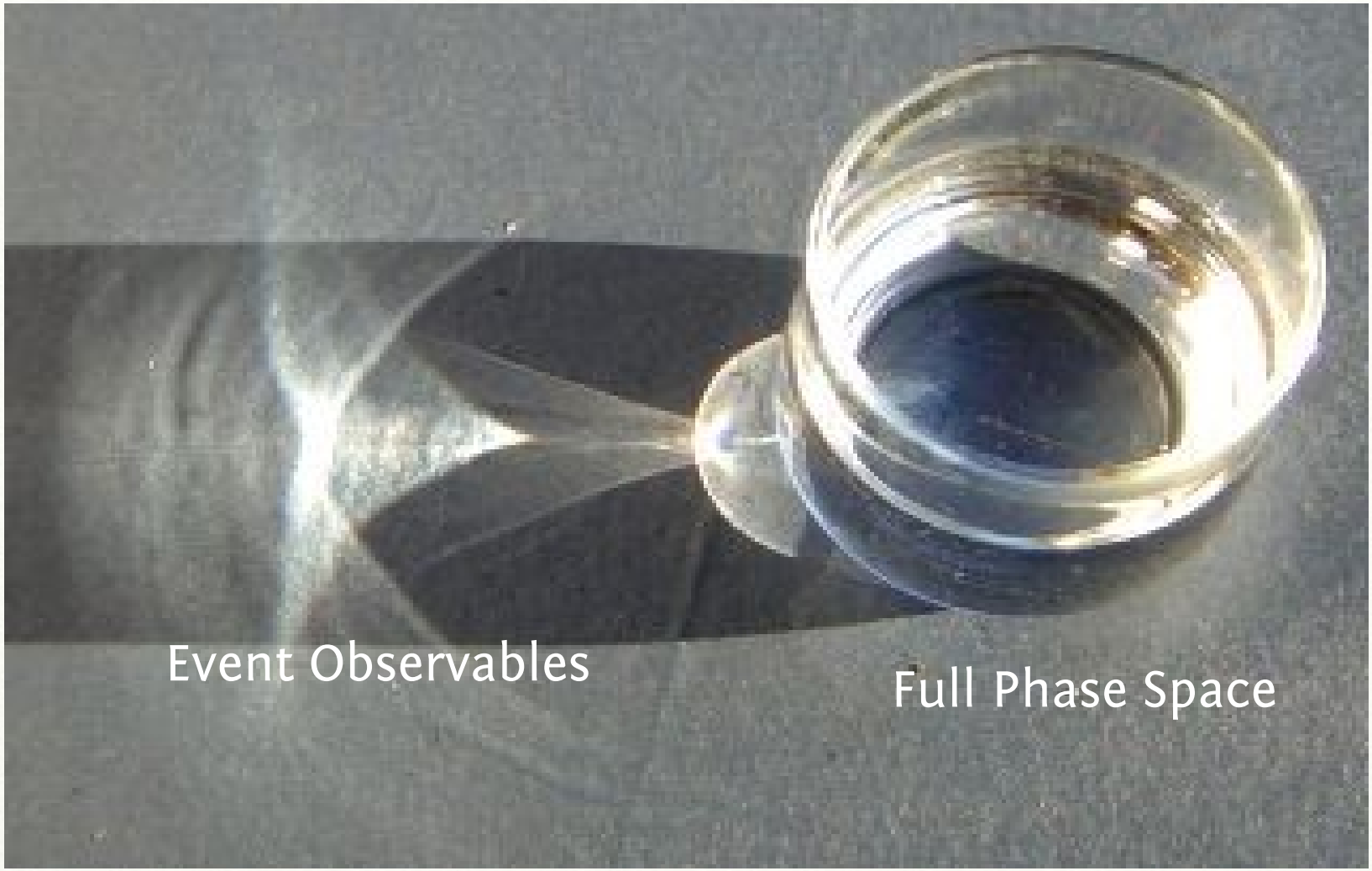
- Phase space is defined by solution space of coupled polynomial equations.

$$\begin{aligned} g_1 &= 0 \\ g_2 &= 0 \\ g_3 &= 0 \\ &\vdots \end{aligned} \longrightarrow \text{affine variety}$$

x : event unknowns q : event observables

- At a singularity point in event observable space,

Jacobian $\left(\frac{\partial g_i}{\partial x_j}\right)$ has a reduced rank.


Event Observables
Full Phase Space

Groebner basis

- With lexicographic ordering $x_1 > x_2 > x_3 > x_4 > \dots$

$$
\begin{aligned}
g_1(x_1, x_2, x_3, x_4, \dots) &= 0 \\
g_2(x_2, x_3, x_4, \dots) &= 0 \\
g_3(x_3, x_4, \dots) &= 0 \\
g_4(x_4, \dots) &= 0
\end{aligned}
$$

- Equations are sequentially solvable.
- Jacobian matrix has upper triangular form.

$$
\left(\frac{\partial g_i}{\partial x_j}\right) = \begin{pmatrix} X & \dots & \dots & \\ & X & X & \dots \\ & & X & \dots \end{pmatrix}
$$

- Vanishing diagonal component is necessary in this basis for a reduced rank of Jacobian.

- For the row vectors of Jacobian with vanishing diagonal component, we can directly check whether they are linearly dependent or not.

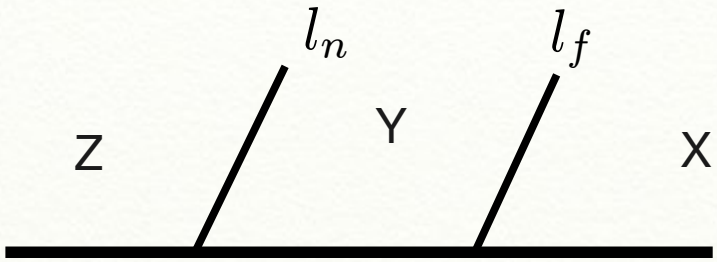
$$\vec{v}_1 = (a_1, a_2, a_3, \dots)$$

$$\vec{v}_2 = (b_1, b_2, b_3, \dots)$$

$$\vec{v}_1 \parallel \vec{v}_2 \quad \text{if} \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots$$

- By this way, we can classify all the singularities in event observables.

EX) Reanalysis of cascade decay



Kinematic constraint equations:

$$X^2 = m_X^2$$
$$(X + l_f)^2 = m_Y^2$$
$$(X + l_f + l_n)^2 = m_Z^2$$

Event unknowns :

$$X^\mu = (X_0, X_1, X_2, X_3)$$

Event Observable :

$$l_n^\mu \qquad l_f^\mu$$

- Coordinate in C.M. frame of l_n and l_f

$$l_n'^{\mu} = \left(\frac{m_{ll}}{2}, 0, 0, \frac{m_{ll}}{2}\right) \qquad l_f'^{\mu} = \left(\frac{m_{ll}}{2}, 0, 0, -\frac{m_{ll}}{2}\right)$$

- Groebner basis : $X_0' > X_3' > X_1' > X_2'$

$$g_1 = 2m_{ll}X_0' + (m_{ll}^2 + m_X^2 - m_Z^2) = 0$$

$$g_2 = 2m_{ll}X_3' + m_{ll}^2 + 2m_Y^2 - m_X^2 - m_Z^2 = 0$$

$$g_3 = m_{ll}^2 X_1'^2 + m_{ll}^2 X_2'^2 + m_{ll}^2 m_Y^2 - (m_Z^2 - m_Y^2)(m_Y^2 - m_X^2)$$

ASM guides the Best Choice of Kinematic Variable

- Jacobian matrix for Groebner basis:

$$\left(\frac{\partial g_i}{\partial X'_j}\right) = \begin{pmatrix} 2m_{ll} & 0 & 0 & 0 \\ & 2m_{ll} & 0 & 0 \\ & & m_{ll}^2 X'_1 & m_{ll}^2 X'_2 \end{pmatrix}$$

- reduced rank Jacobian arise when 3rd row vanishes

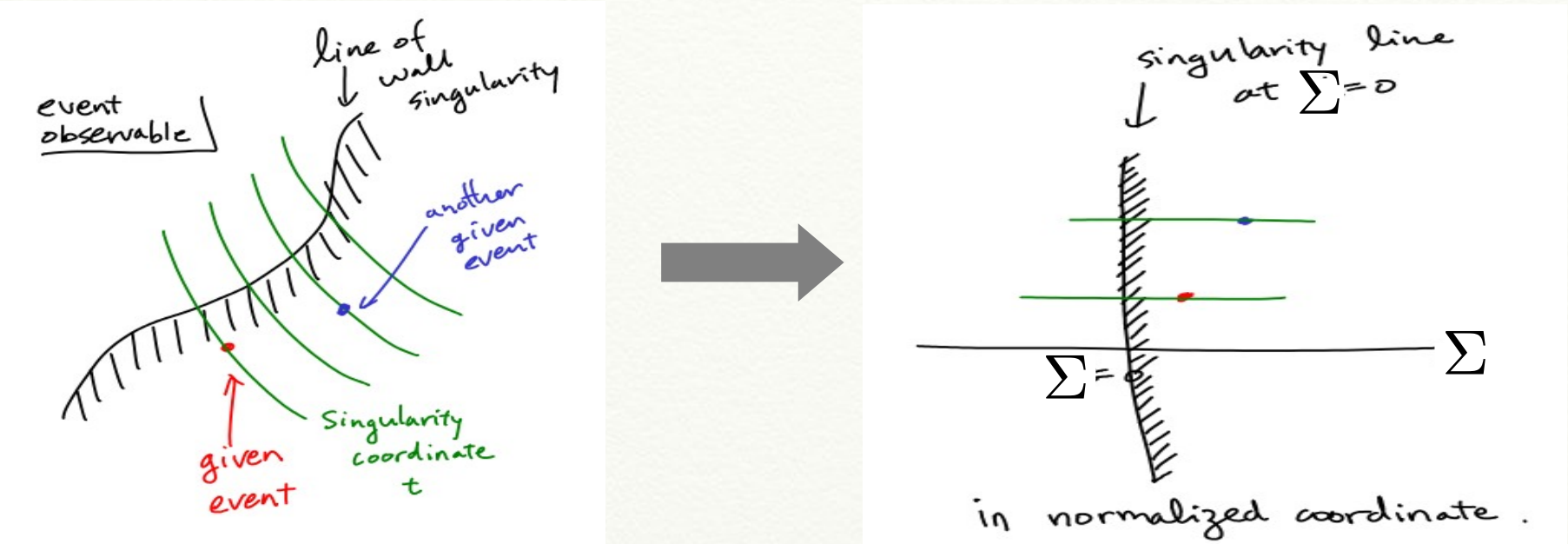
$$m_{ll}^2 = \frac{(m_Z^2 - m_Y^2)(m_Y^2 - m_X^2)}{m_Y^2} \equiv m_{ll}^{(\mathrm{end})\,2}$$

- Note that we did not start from the invariant mass. Algebraic Singularity Method shows that the invariant mass has singularities.

Singularity Coordinate

1. Singularities must collectively appear on the same point in the singularity coordinate.

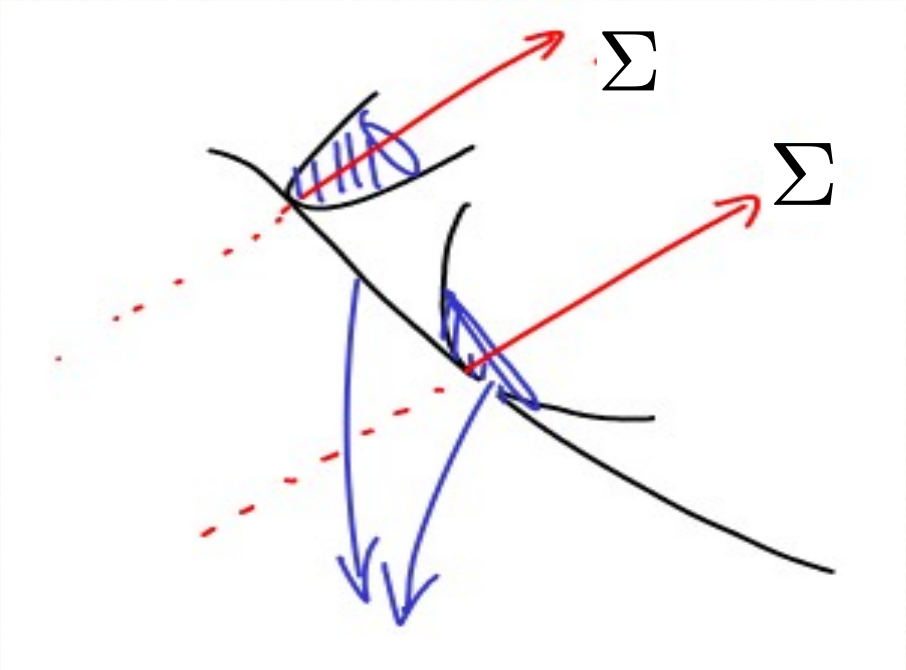
2. Singularities must be perpendicular to the singularity coordinate direction.

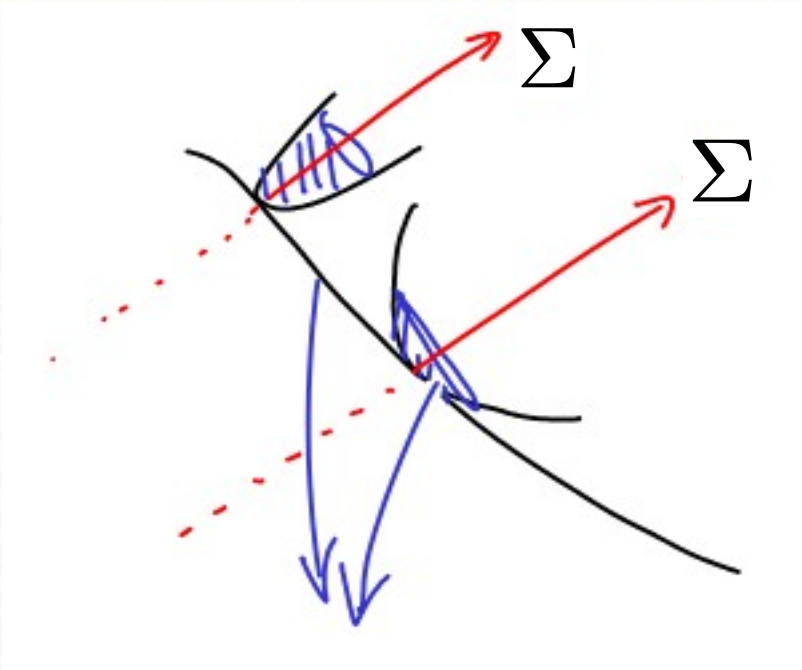


- Singularity coordinate direction is already determined by the reduced rank condition.

$$\frac{\partial g}{\partial(q,x)} = \begin{array}{c} \begin{array}{cc} \frac{\partial g}{\partial q} & \frac{\partial g}{\partial x} \end{array} \\ \left(\begin{array}{c|c} \times\ \times\ \times\ \times & \times\ \times\ \times\ \times \\ \hline \boxed{\vec{v}} & \boxed{0\ 0\ 0\ 0\ 0\ \cdots} \\ \hline \times\ \times\ \times\ \times & \times\ \times\ \times\ \times \\ \times\ \times\ \times\ \times & \times\ \times\ \times\ \times \end{array}\right) \end{array}$$

3. Each singularity must have the same statistical significance.





- Local description of phase space is useful for this procedure :

  second fundamental form

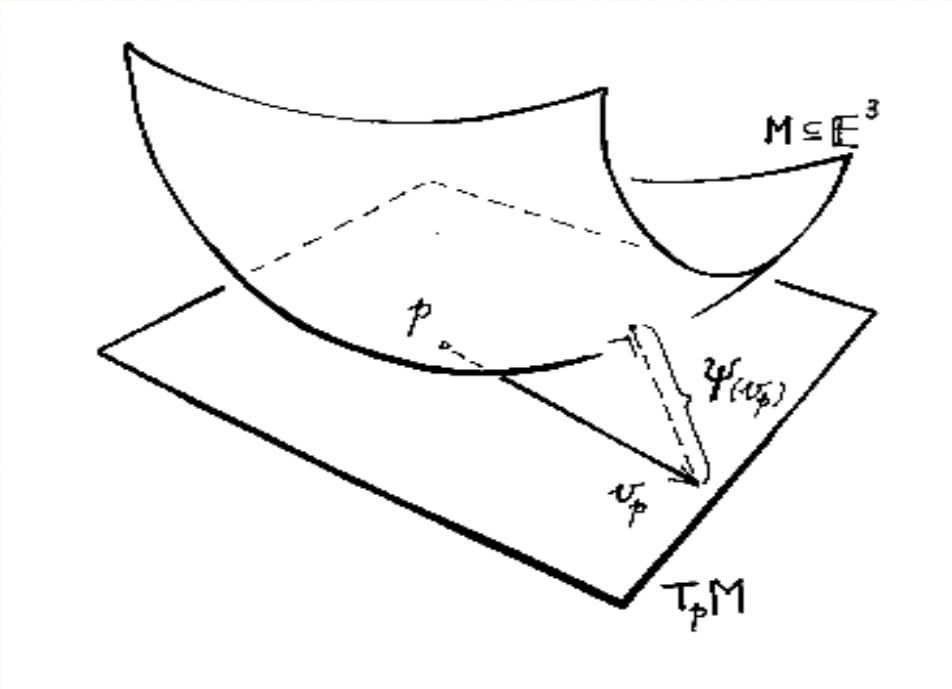
- Same invisible momentum volume gives rise to the same singularity coordinate.

Using the Second Fundamental Form

(Extrinsic Curvature Form)

- Describe 2nd order discrepancy between tangential plane and actual hypersurface.

$$I\!I(v,w) = (\nabla_v w)^{\perp}$$ (curvature information)



Second Fundamental Form of Algebraic Variety

$$g_i = 0 \quad \Rightarrow \quad g_i + \frac{\partial g_i}{\partial y_j} dy^j + \frac{\partial^2 g_i}{\partial x_j \partial x_k} dx^j dx^k = 0$$

local variation

Tangential direction

Normal direction

$$\frac{\partial g_i}{\partial y_j} dy^j = -\frac{\partial^2 g_i}{\partial x_j \partial x_k} dx^j dx^k$$

Defines a quadratic map : Tangent → Normal

Condition 1,2 and 3 turn out nothing but 0th , 1st and 2nd order boundary conditions at singular points.

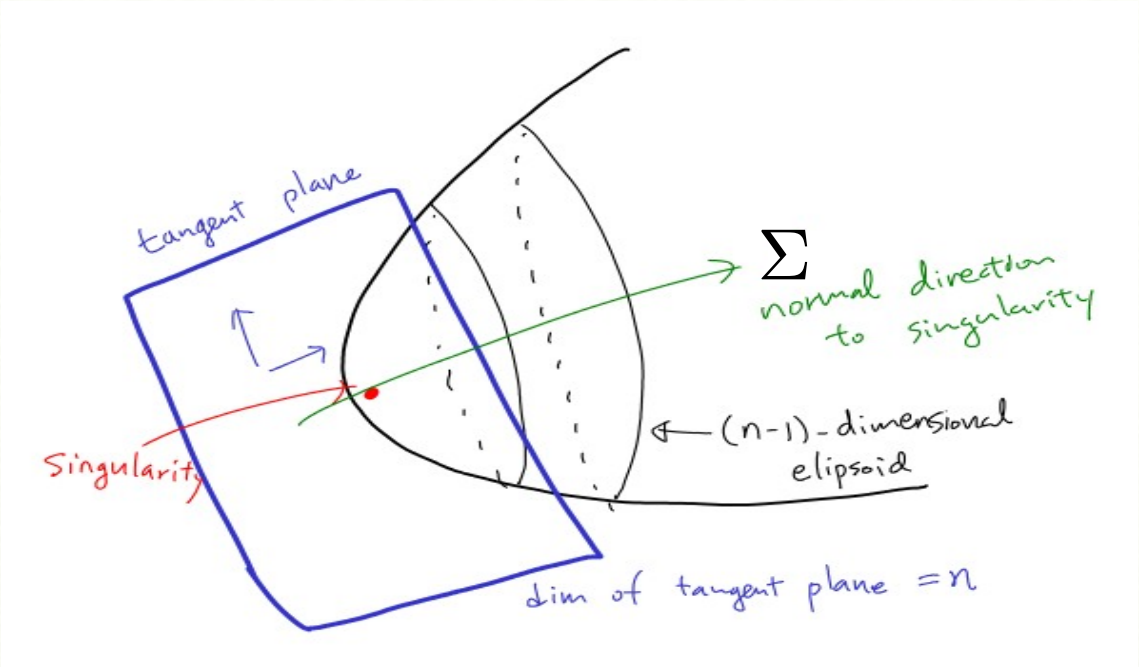
Normalization and Local scaling near singularity

- Define singularity coordinate (up to a scale factor)

$$\Sigma \propto \vec{v} \cdot I\!I$$

- The linear coordinate is characterized by (n x n) matrix M, where n is the dimension of tangent plane immersed completely along invisible momentum space.

- We have to choose scale factor of Σ so to have the same statistical weight for each singularity point.



Locally,

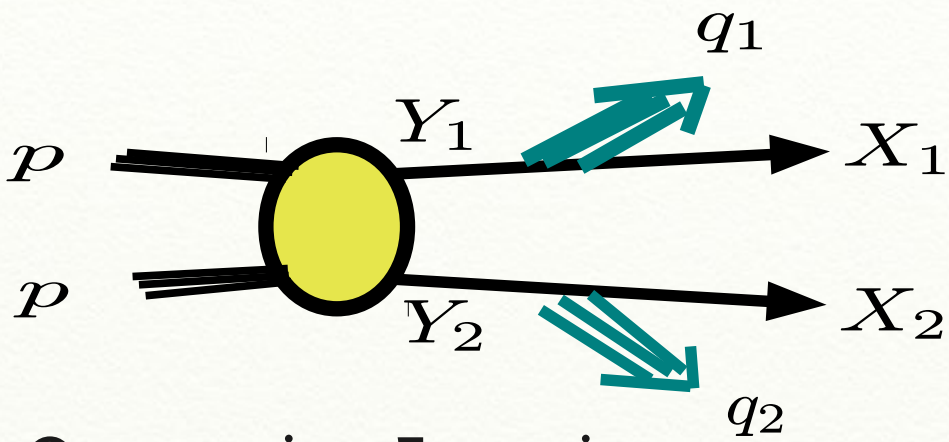
$$\tilde{\Sigma} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2}$$

$$(\mathrm{vol}) \propto (a_1 a_2 \cdots a_M)^{-1/2} \, \hat{\Sigma}^{M/2}$$

$$\Sigma \equiv (a_1 \ldots a_M)^{-1/M} \, \tilde{\Sigma}$$

$$\frac{d\Gamma}{d\Sigma} \sim \Sigma^{\frac{M-2}{2}} \frac{d\Gamma}{d(\mathrm{Vol})}$$

Ex) Double Missing Particle Chain Topology



Kinematic Constraint Equations :

$$X_1^2 = m_X^2 \qquad X_2^2 = m_X^2$$

$$(X_1 + q_1)^2 = m_Y^2 \qquad (X_2 + q_2)^2 = m_Y^2$$

$$\vec{X}_{1T} + \vec{X}_{2T} = \vec{P}_T^{\text{miss}}$$

Event unknowns:

$$X_1^\mu = (X_{10}, X_{11}, X_{12}, X_{13}) \qquad X_2^\mu = (X_{20}, X_{21}, X_{22}, X_{23})$$

Event observables:

$$q_1^\mu = (q_{10}, q_{11}, q_{12}, q_{13}) \quad q_2^\mu = (q_{20}, q_{21}, q_{22}, q_{23}) \quad \vec{P}_T^{\text{miss}} = (P_{T1}, P_{T2})$$

Groebner basis : $X_{10} > X_{13} > X_{20} > X_{21} > X_{22} > X_{23} > X_{11} > X_{12}$

$$
\begin{aligned}
g_1 &= \left(q_{20}^2 - q_{23}^2\right)\chi_{23}^2 + 2q_{21}q_{23}\chi_{23}\chi_{11} + 2q_{22}q_{23}\chi_{23}\chi_{12} - 2C_2q_{23}\chi_{23} \\
&\quad + \left(q_{20}^2 - q_{21}^2\right)\chi_{11}^2 - 2q_{21}q_{22}\chi_{11}\chi_{12} + \left(q_{20}^2 - q_{22}^2\right)\chi_{12}^2 \\
&\quad + \left(-2P_{T1}q_{20}^2 + 2C_2q_{21}\right)\chi_{11} + \left(-2P_{T2}q_{20}^2 + 2C_2q_{22}\right)\chi_{12} \\
&\quad + \left(\vec{P}_T^2 + m_\chi^2\right)q_{20}^2 - C_2^2, \\
g_2 &= \chi_{22} + \chi_{12} - P_{T2}, \\
g_3 &= \chi_{21} + \chi_{11} - P_{T1}, \\
g_4 &= q_{20}\chi_{20} - q_{23}\chi_{23} + q_{21}\chi_{11} \\
&\quad + q_{22}\chi_{12} - C_2, \\
g_5 &= \left(q_{10}^2 - q_{13}^2\right)\chi_{13}^2 - 2q_{11}q_{13}\chi_{13}\chi_{11} - 2q_{12}q_{13}\chi_{13}\chi_{12} - 2C_1q_{13}\chi_{13} \\
&\quad + \left(q_{10}^2 + q_{11}^2\right)\chi_{11}^2 + 2q_{11}q_{12}\chi_{11}\chi_{12} \\
&\quad + \left(q_{10}^2 + q_{12}^2\right)\chi_{12}^2 - 2C_1q_{11}\chi_{11} - 2C_1q_{12}\chi_{12} \\
&\quad + \left(m_\chi^2q_{10}^2 - C_1\right), \\
g_6 &= q_{10}\chi_{10} - q_{13}\chi_{13} - q_{11}\chi_{11} - q_{12}\chi_{12} - C_1,
\end{aligned}
$$

Solution as a function of χ_{11} and χ_{12}

$$\chi_{23}^{\rm soln} = \frac{A_2 q_{23} \pm q_{20}\sqrt{A_2^2 - \left(q_{20}^2 - q_{23}^2\right)\left(m_\chi^2 + \vec{k}_{2T}^2\right)}}{q_{20}^2 - q_{23}^2},$$

$$\chi_{22}^{\rm soln} = P_{T2}^{\rm miss} - \chi_{12},$$

$$\chi_{21}^{\rm soln} = P_{T1}^{\rm miss} - \chi_{11},$$

$$\chi_{20}^{\rm soln} = \frac{A_2 + q_{23}\chi_{23}}{q_{20}},$$

$$\chi_{13}^{\rm soln} = \frac{A_1 q_{13} \pm q_{10}\sqrt{A_1^2 - \left(q_{10}^2 - q_{13}^2\right)\left(m_\chi^2 + \vec{k}_{1T}^2\right)}}{q_{10}^2 - q_{13}^2},$$

$$\chi_{10}^{\rm soln} = \frac{A_1 + q_{13}\chi_{13}}{q_{10}},$$

$$A_1 = C_1 + \vec{q}_{1T}\cdot\vec{\chi}_{1T},$$

$$A_2 = C_2 - \vec{q}_{2T}\cdot\vec{\chi}_{1T}.$$

Jacobian Matrix :

$$
\begin{array}{c}
g_6 \\ g_5 \\ g_4 \\ g_3 \\ g_2 \\ g_1
\end{array}
\begin{pmatrix}
X & X & & & & & X & X \\
 & X & & & & & X & X \\
 & & X & & & X & X & X \\
 & & & 1 & & & 1 & \\
 & & & & 1 & & & 1 \\
 & & & & & X & X & X
\end{pmatrix}
$$

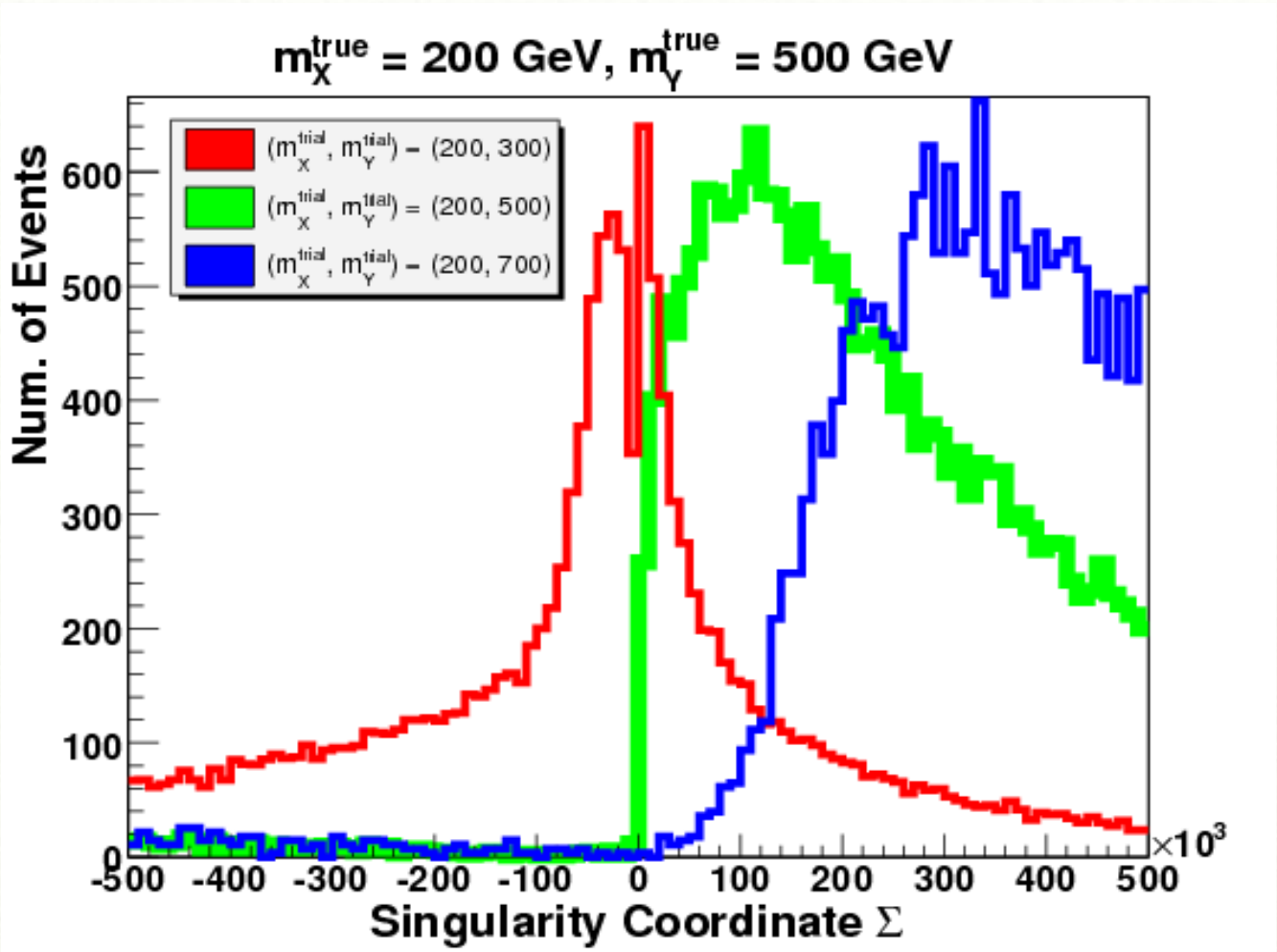
reduced rank condition :

$$
\left.
\begin{aligned}
2(q_{10}^2 - q_{13}^2)X_{13} - 2A_1 q_{13} &= 0 \\
2(q_{20}^2 - q_{23}^2)X_{23} - 2A_2 q_{23} &= 0
\end{aligned}
\right\}
$$

$$
\det \begin{pmatrix} \frac{\partial g_5}{\partial X_{11}} & \frac{\partial g_5}{\partial X_{12}} \\ \frac{\partial g_1}{\partial X_{11}} & \frac{\partial g_1}{\partial X_{12}} \end{pmatrix} = 0
$$

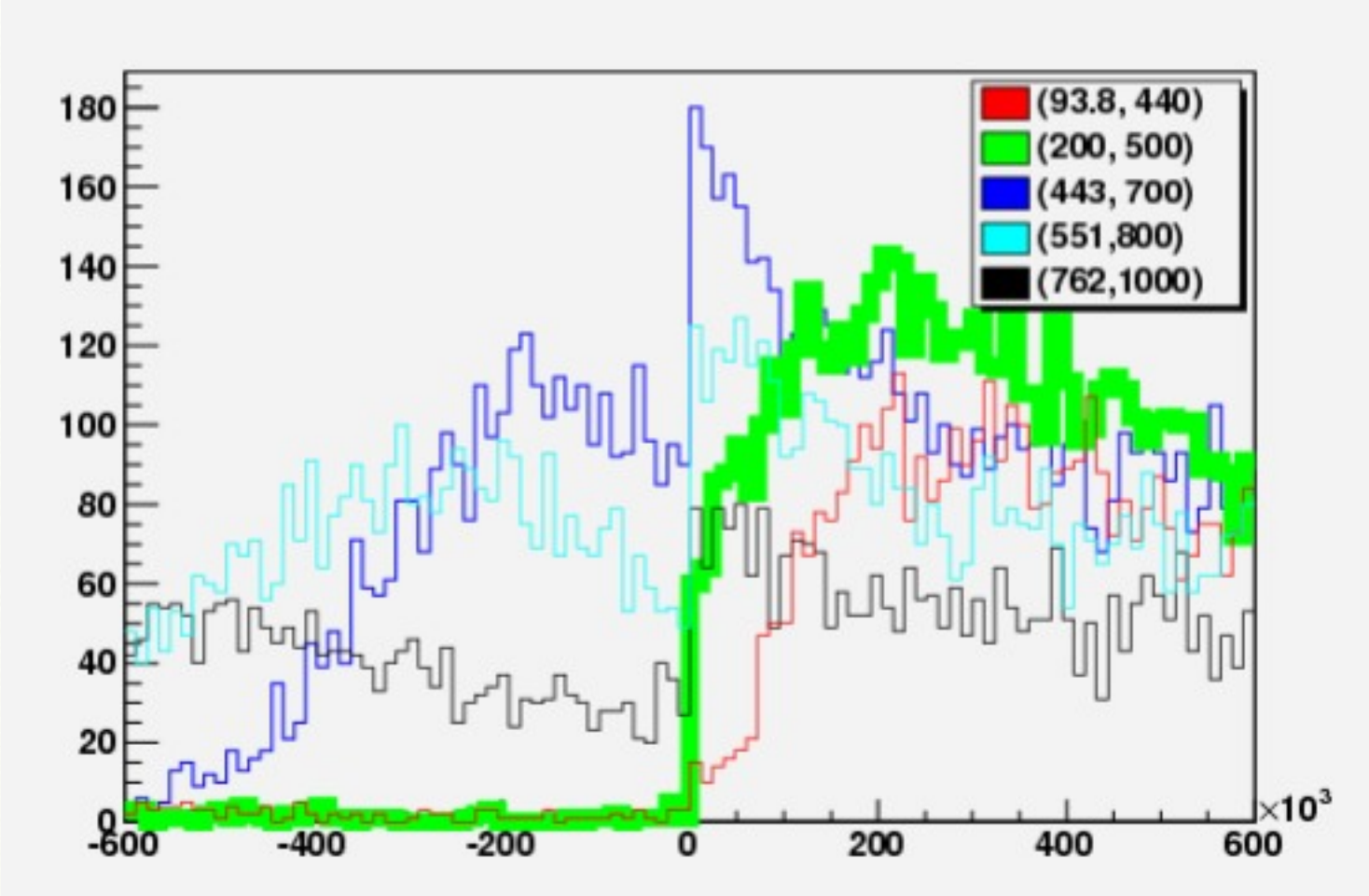
Yields the maximum MT2 condition and MAOS momentum.

Numerical Analysis



Singularity coordinate distribution along mT2 max curve.

IWK, work in progress



Conclusion

- Key Idea in mass measurement technique:
  PS folding → Singularities

- A new method (Algebraic Singularity Method) developed for finding PS singularities and constructing optimized kinematic variable that exploits the singularities

- Naturally encompasses previous methods for non-reconstructable event topologies and can be generalized to more complicated event topologies.

Thank you!