

BAYESIAN DETECTION ANALYSIS FOR RADIATION EXPOSURE

G. Miller, W. C. Inkret, and H. F. Martz

Los Alamos National Laboratory, Los Alamos, NM 87545, USA

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Abstract: A new analysis method for determining whether a measured bioassay, dosimetry, environmental monitoring, or other result should be called 'positive' or 'zero' is proposed, based on the principles of Bayesian inference. Bayesian methods permit the incorporation of prior knowledge. If, for example, a facility historically has a very small number of real positive results, then a new measured result must be a greater number of standard deviations away from zero in order to imply that the new result is actually positive. Prior knowledge can have an important effect on the interpretation of measurements. Classical decision theory of detection limits does not take prior knowledge into account.

INTRODUCTION

A situation commonly encountered in the interpretation of bioassay measurements is illustrated by the following example. A worker, Mr Smith, is on a routine bioassay programme. A bioassay measurement is performed and gives 0.03 ± 0.01 in some units. Mr Smith sees the result and is upset. 'I'm very concerned about radiation and I really want to know if I've gotten any radioactivity in me.' Usually Mr Smith is given a simplified review of measurement statistics and classical decision theory. We tell him a lot about the bioassay measurement. In most cases Mr Smith's question is not answered.

The problem posed by this example was described in 1962 by John Tukey, 'Far better an approximate answer to the right question, which is often vague, than an exact answer to the wrong question, which can always be made precise'⁽¹⁾.

Using the principles of Bayesian inference, there is a method for providing reasonable answers to the questions posed by Mr Smith. After applying this method, which is described here, we would say to Mr Smith 'given what we know about your work situation, the probability you have work-related radioactivity in you is 10%'. A detection result could be interpreted as 'positive' if the probability were greater than some limiting probability, say 50%. If the probability of work-related activity were less than the limiting probability, the measurement result would be interpreted as 'zero'. Using this definition of positive and zero in Smith's case, we could simply say 'your measurement indicated zero'. The precise meaning of these statements will be given in what follows. Results interpreted in this fashion may be better understood by Mr Smiths, managers, the general public, or in short all those who want bottom-line answers to the questions of

real interest.

This simple, intuitive, concept of a 'positive' or 'zero' result would be useful in *in vivo* and *in vitro* bioassay, external radiation dosimetry, environmental monitoring, and other areas, particularly for sample populations where the fraction of true positive results is small and the number of samples is large. The method is also useful for interpreting limited or partial measurements, before there is an adequate amount of new measured information under classical decision theory.

Altshuler and Pasternack⁽²⁾ and Currie⁽³⁾ have discussed classical, non-Bayesian, detection limits based on statistical tests of Poisson populations⁽⁴⁻⁷⁾. As will be shown, the Bayesian concept of a positive detection result cannot be directly related to these concepts because of fundamental differences in the concepts. The difference between Bayesian detection limits and classical detection limits becomes larger when the population of interest has a smaller fraction of real positive results. In that case, a measured result needs to be a greater number of standard deviations away from zero in order to imply a real positive result.

Bayesian and classical methods differ in both practical and philosophical aspects. The major practical distinction is that Bayesian methods permit the formal incorporation of prior knowledge, belief, and information beyond that contained in the observed data in the inference process. This additional information may range from virtually complete ignorance, through intermediate knowledge, or the additional information may be strongly informative as for the example discussed in this paper. This additional information is embodied in the so-called prior distribution.

Bayesian methods of statistical inference are

described in several textbooks⁽⁸⁻¹⁴⁾. The past two decades have witnessed an explosion in the use of Bayesian methods, particularly in the physical sciences (see ch. 14 of Ref. 12). An excellent short review of the subject is given by Sivia⁽¹⁵⁾.

Bayesian analysis has been discussed in the health physics literature by Little⁽¹⁶⁾, who considered a counting situation where a background count is subtracted. The measured result can then be negative, whereas the true result is known with certainty to be non-negative. Using a prior distribution with zero probability for negative results changes the estimated result from that given by classical methods. Strom⁽¹⁷⁾ comments on Little's paper that it is not necessarily known that the net counting rate must be positive. However the true quantity of interest (e.g. activity, concentration) must be non-negative. A possible systematic error of background subtraction can be included in the measurement error.

MATHEMATICAL MODEL

The true (but unknown) quantity of concern will be denoted by y_0 . This could be the activity of plutonium in urine, for example. There is always a natural background, which may be very small (everyone has at least a few atoms of plutonium in them, for example). We assume a natural background level of y_0 that is very small. Figure 1 shows an example of the probability density of finding amount y_0 (in interval dy_0), for people in the general population without any work-related exposure potential. The quantity y_{0c} is a critical level chosen such that almost no one in the general population has $y_0 > y_{0c}$. The measurement result is denoted by y , with measurement error Δy . In the general case, the measured value y has some known (or postulated) relationship to the true value y_0 . In this paper, we assume that

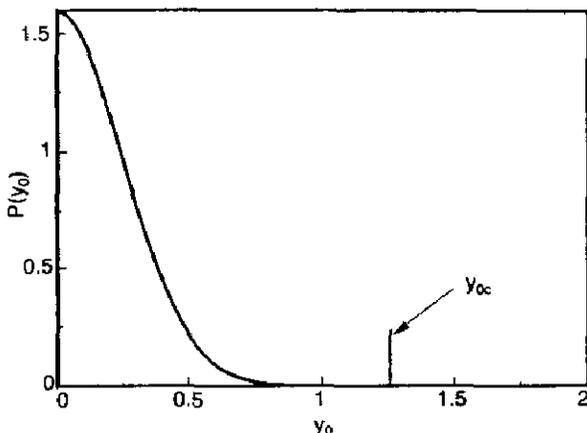


Figure 1. Background distribution of y_0

$y = y_0 + \Delta y$ where Δy is a random measurement error with mean 0 and standard deviation $\sigma(y_0)$. We imagine a situation where the natural background is much smaller than the measurement error, so that $y_{0c} \ll \sigma(y_0 = 0)$. The precise value of y_{0c} is unimportant as long as the following conditions hold true: (1) almost no one in an uncontaminated population has $y_0 > y_{0c}$ and (2) $y_{0c} \ll \sigma(y_0 = 0)$

The statement 'the result was positive' is defined to mean that the probability of a positive result is greater than the limiting probability, which is denoted by $1-\gamma$. Mathematically, this statement becomes

$$P(y_0 > y_{0c} | y) = \int_{y_{0c}}^{\infty} P(y_0 | y) dy_0 > 1-\gamma \quad (1)$$

(The symbol $P(A|B)$ denotes the probability of A given that B is true. We distinguish by context probability densities and true probabilities. Thus, if A is a continuous variable, $P(A)$ needs to be multiplied by dA to become the probability that A is in the range $A-dA/2$ to $A+dA/2$.)

In words, Equation 1 reads: The probability that the true result y_0 is greater than the critical level y_{0c} given measurement result y , which is equal to the integral over y_0 from y_{0c} to ∞ of the probability $P(y_0 | y) dy_0$ that y_0 is in interval dy_0 given y , is greater than the limiting probability $1-\gamma$. Equivalently, since $P(y_0 > y_{0c} | y) + P(y_0 \leq y_{0c} | y) = 1$

$$P(y_0 \leq y_{0c} | y) = \int_0^{y_{0c}} P(y_0 | y) dy_0 < \gamma \quad (2)$$

This definition of 'the result was positive' depends on a choice of γ . In this paper we will choose $\gamma = 0.5$. This definition also appears to depend on the critical level y_{0c} , but it does not as long as the two requirements discussed above are satisfied.

It is important to appreciate the difference between talking about y_0 , the true (but unknown) quantity, and y , the measured quantity. We are really interested in estimating y_0 . The measurement result y is of interest only because of its value in estimating y_0 . By contrast, classical discussions of detection limits (e.g. Ref. 3) focus on y , given some assumptions about y_0 . But no matter how precise and intricate such a discussion is, it will not answer the real question of interest: What is the probability distribution of y_0 given the new information contained in the measurements and information we knew beforehand about the situation? Only the Bayesian method can address this question.

Given the true quantity y_0 , the measurement result is assumed to have a Gaussian distribution,

$$P(y|y_0) = \frac{1}{\sqrt{2\pi}[\sigma(y_0)]^2} \exp\left\{-\frac{(y-y_0)^2}{2[\sigma(y_0)]^2}\right\} \quad (3)$$

for $-\infty < y < \infty$ with measurement error standard deviation $\sigma(y_0)$. In many cases of interest the assumption of Gaussian measurement errors is a good approximation.

Given Equation 3, how do we evaluate condition 1 or 2? This is where Bayes formula comes in. Bayes formula (see Appendix 1) states

$$P(y_0|y) = C(y)P(y|y_0)P(y_0) \quad (4)$$

where $C(y)$ is a normalisation constant (constant with respect to y_0), $P(y|y_0)$ is given by Equation 3, and $P(y_0)$ is the probability distribution of y_0 that we would estimate prior to the measurement (the prior distribution). The normalisation constant $C(y)$ is, in fact, the reciprocal of the unconditional probability distribution $P(y)$ of the measurement, which is sometimes called the evidence (see Appendix 1).

What is $P(y_0)$? This depends on our prior knowledge of relevant events or information. The information could include the type of facility, quantities of material in use, historical records of previous bioassay results, and information about specific incidents in which an individual has been involved. In what follows a rather simple model will be used, but this can obviously be made more complicated and/or accurate.

From the previous history of the facility we believe that some fraction ϵ of all workers are positive (i.e. have amounts of radioactivity in their bodies above the background level). A procedure for estimating ϵ is discussed in Appendix 2. The distribution of the true quantity y_0 for positive workers is assumed to be uniform up to a maximum amount y_{0max} . Because the background is essentially zero on the scale determined by measurement error, the distribution of y_0 for the 'zero' class of people is taken as a delta function. Thus, we assume

$$P(y_0) = (1 - \epsilon)\delta(y_0) + \frac{\epsilon}{y_{0max}} \quad (5)$$

for $0 \leq y_0 < y_{0max}$ (The delta function $\delta(x)$ is the limit of functions peaked at $x = 0$ with unity integral such that in the limit, $\delta(x) = 0$ for $x \neq 0$ and $\int \delta(x)dx = 1$.)

The prior given by Equation 5 is used as an example only. Those who are serious about using this method should construct priors based on knowledge and experience of their own situations. The uniform distribution of positive y_0 values is not likely to be realistic. It is used because (1) it is simple, and (2) it is conservative, in that it skews the calculated probability distribution of y_0 toward

larger values, relative to a more realistic peaked functional form.

The prior given by Equation 5 has two parameters: y_{0max} and ϵ . Using this prior, an analytical solution for $P(y_0|y)$, the probability distribution of true quantity y_0 given the results of one or more measurements, is given in Appendix 3. The formulas in Appendix 3 can readily be coded up in a simple computer program to calculate whatever quantities might be of most interest.

NUMERICAL EXAMPLE

Consider the following numerical example concerning *in vivo* lung counts evaluated for the presence of ²⁴¹Am. We assume $y_{0max} = 1.7$ nCi, $\epsilon = 0.005$ (0.5% of people belong to the 'positive' class, and, without additional information, we expect those people to have a uniform distribution of lung burdens extending from 0 to 1.7nCi), and $\gamma = 0.5$.

Equation 3 is found to be a good model for the measurement errors for the Los Alamos *in vivo* counting system. A large subtraction of body background is involved, which gives the possibility of negative measured results. The measurement error standard deviation is found to be of the form

$$\sigma(y_0) = \sigma_0\sqrt{(1 + y_0/a)} \quad (6)$$

with $\sigma_0 = 0.1$ nCi and $a = 0.4$ nCi. Equation 6 was obtained from a linear fit of observed variances to measured results.

The commonly used decision level or critical level L_c is the level that gives some specified rate of false positives⁽³⁾. For 5% false positives $L_c = 1.645\sigma(y_0 = 0) = 1.645\sigma_0 = 0.165$ nCi. Let us assume that the measurement gave $y = L_c$. Even though L_c is precisely defined, it is tempting to assume that if the false positive rate is 5%, 95% of the time a result $\geq L_c$ would be actually positive. This would be incorrect! For our example, after a measurement giving $y = L_c$, Equation 1 and Equation A3.5 in Appendix 3 show the probability that the result is positive is only 0.3%! This numerical example is discussed further in the next section, and the result can be obtained from Figure 4. If the measurement is repeated and the average is greater than L_c , the probability of a positive result is still only 0.6%. Only after seven repetitions of the measurement does the average value being greater than L_c imply at least a 50% probability that the result is positive.

For a single measurement to imply a positive result, it has to be greater than 0.38nCi (see Figure 4). For two measurements, the average must be greater than 0.28nCi.

ALTERNATIVE DERIVATION

Let us assume that we have a group of N people having a distribution of y_0 given by Equation 5, so that

$$\frac{dN}{dy_0} = NP(y_0) \tag{7}$$

All of these people are measured and a distribution of results y is calculated. The probability of measurement result y in interval dy is given in terms of $P(y_0)$ by the equation

$$P(y) = \int_0^\infty P(y|y_0) P(y_0) dy_0 \tag{8}$$

and

$$\frac{dN}{dy} = NP(y) \tag{9}$$

Equation 8 results from the fact that the event {'measured y '} is the union of the non-intersecting events {'true amount y_0 '} AND {'measured y given y_0 '} over all possible values of y_0 .

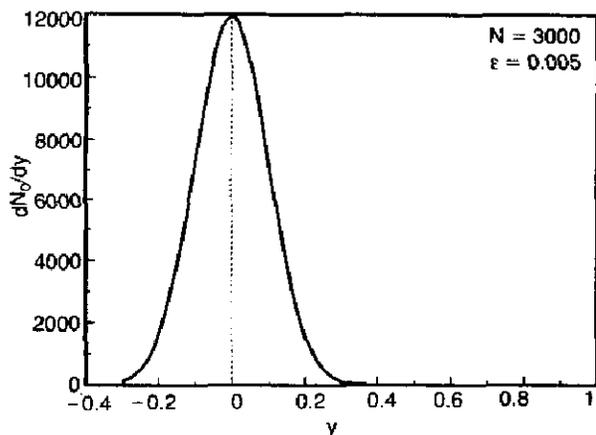


Figure 2. Distribution of measured results from clean people.

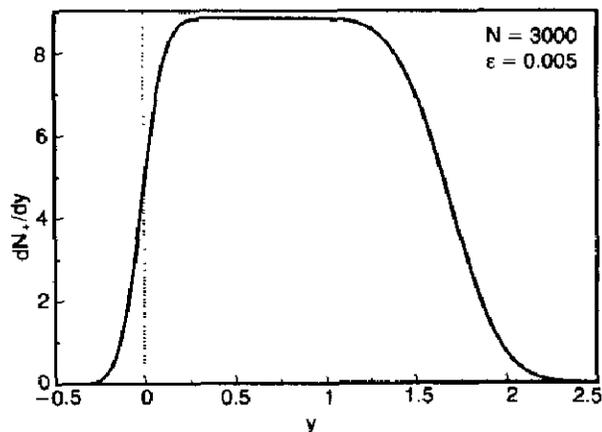


Figure 3. Distribution of measured results from positive people.

Let us separate dN/dy into two components, (1) results from people in the 'zero class' dN_0/dy , and (2) results from people in the 'positive class' dN_+/dy . The quantity dN_0/dy is calculated using Equation 9 with $P(y_0) = (1-e)d(y_0)$ and dN_+/dy is calculated using Equation 9 with $P(y_0) = e/y_0$ for $y_0 < y_{0max}$. Figures 2 and 3 show dN_0/dy and dN_+/dy for a sample N of 3000 people using the parameters of the numerical example already discussed. Figure 2 shows the distribution of measured results from $3000(1-e) = 2985$ people that belong to the 'zero' class, so it just reflects the measurement error around zero. Figure 3 is a plot of the measured results from $3000e = 15$ people in the 'positive' class, if the distribution of y_0 takes the form given by Equation 5. The integrals under the curves in Figures 2 and 3 are 2985 and 15 respectively, corresponding to the numbers of people in the 'zero' and 'positive' classes.

The distribution of measured results is a sum of dN_0/dy from Figure 2 and dN_+/dy from Figure 3. For a given measured value y , the fraction of results that are zero is

$$f_0 = \frac{dN_0/dy}{dN_0/dy + dN_+/dy} \tag{10}$$

Equation 10 is exactly the same as the Bayesian probability $P(y_0 \leq y_0c|y)$ calculated using Equation 2, and Equation A3.5 in Appendix 3. The fraction of results that are positive is given by

$$f_+ = 1 - f_0 \tag{11}$$

which is the same as the Bayesian probability of a positive result given by Equation 1. Figure 4 shows f_+ plotted against measured value y (for a single measurement) for three different values of ϵ (using the parameters of the numerical example already discussed).

For very small values of ϵ as in Figures 2 and 3, dN_0/dy is much larger than dN_+/dy . This means

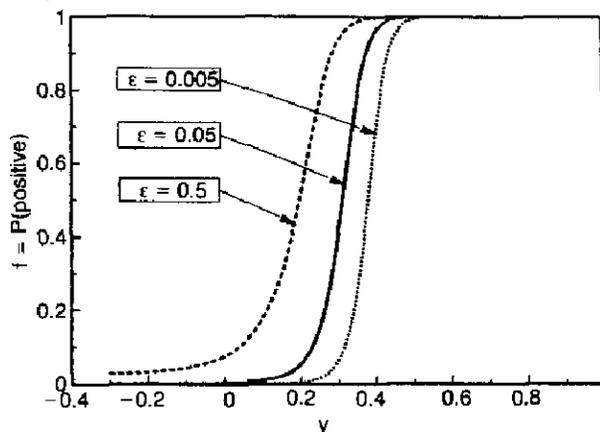


Figure 4. Probability of a positive result given measured result y .

that the measured result y , even if fairly large, is likely to be just the tail of the distribution of people with zero true amount. Eventually this tail becomes exponentially small and a large enough measured y implies a positive result. Figure 4 shows how the probability that the result is positive depends on both ϵ , the parameter giving the number of true positives in the prior distribution, and y , the measured result.

DISCUSSION

In a clean facility (or sample population) where only a small fraction of cases are truly positive, the measured results must be a larger number of standard deviations away from zero before the result can be interpreted as positive. This is an important result from the Bayesian analysis. The Bayesian analysis does require us to estimate the prior distribution; there is no logical way to avoid doing this. The assumption of a uniform prior distribution, which is often implicitly made, is a misleading approximation in many cases. The Bayesian approach forces us to state our assumptions clearly, and provides a method for quantifying subjective methods for doing the same thing that are often used in practice.

APPENDIX 1

Bayes Formula

For convenience, a heuristic derivation of Bayes formula will be given here. The formula for condition probability is

$$P(A|B) = P(AB)/P(B) \tag{A1.1}$$

where $P(A|B)$ is the probability of A given B while $P(AB)$ is the probability of A and B. Interchanging A and B,

$$P(B|A) = P(AB)/P(A) \tag{A1.2}$$

Therefore,

$$P(B|A) = P(A|B)P(B)/P(A) \tag{A1.3}$$

which is Bayes formula

APPENDIX 2

Estimating the Prior Distribution

To estimate the prior distribution, we start with a set of results that logically pertain to the case in question. The unconditional distribution of measurement results from this data set is given, in terms of $P(y_0)$, by

$$P(y) = \int_0^\infty P(y|y_0)P(y_0)dy_0 \tag{A2.1}$$

The procedure described in this paper can be used with other, less crude, methods of estimating the prior distribution using other information. One approach is to use empirical Bayes method⁽¹⁴⁾. Using this approach, a relative frequency-based prior distribution is determined from the past measurement results at the facility, similar to what was done here, but in a more formal way.

In estimating the prior distribution using historical data as done in this paper, we have assumed that what has happened in the past is applicable to the present. This is usually a conservative assumption for radiation safety, since the technology and methods are improving with time.

There are other alternative Bayesian approaches to the decision problem considered here. It can be formulated as a 2×2 Bayesian decision analysis problem in which the two states of nature are $\{\theta_1, \theta_2\} = \{y_0 \leq y_{0c}, y_0 > y_{0c}\}$ that is, {true 'Zero,' true 'Positive'}, and the two actions are $\{a_1, a_2\} = \{\text{conclude 'Positive,' conclude 'Zero'}\}$ (see Ref. 13, Sec. 5.4). A major difficulty in using this approach is the necessity of considering and specifying a 2×2 loss function representing the loss incurred when action a_1 is taken and the state of nature is the θ_1 .

Knowing $P(y|y_0)$ and $P(y)$, it is theoretically possible using various techniques to estimate $P(y_0)$ from Equation A2.1. In our two parameter representation, $P(y_0)$ is given by Equation 5.

The problem in estimating ϵ occurs because the portion of the distribution arising from the delta function term is much larger than the ϵ term. A very simple method will be used here. The quantity y_{0max} is taken to be the largest value of y in the data set. Then, starting with some assumed ϵ , we find the value y_1 of y such that a single measurement implies a positive result. Then we see how many results N_1 in the data set have $y > y_1$. Finally, ϵ is obtained from the formula

$$N_1 = \epsilon N \frac{y_{0max} - y_1}{y_{0max}} \tag{A2.2}$$

where N is the total number of results in the data set. This process is iterated until it converges to a final value of ϵ .

APPENDIX 3

Analytical Solution for $P(y_0|y)$

After N independent measurements giving results y_1, y_2, \dots, y_N

$$P(y_0|y_1, y_2, \dots, y_N) = C \prod_{i=1}^N P(y_i|y_0)P(y_0) \quad (A3.1)$$

where the normalisation constant C is independent of y_0 . Substituting Equation 3 for $P(y_i|y_0)$ into Equation A3.1

$$P(y_0|y_1, y_2, \dots, y_N) = C \frac{1}{[\sigma(y_0)]^N} \exp\left\{-\sum_{i=1}^N \frac{(y_i - y_0)^2}{2[\sigma(y_0)]^2}\right\} P(y_0) \quad (A3.2)$$

Some factors are absorbed into C to make a new normalisation constant C'.

The sum in the exponential in Equation A3.2 can be written as

$$\sum_{i=1}^N (y_i - y_0)^2 = N(\bar{y}^2 - 2y_0\bar{y} + y_0^2) = N(y_0 - \bar{y})^2 + N(\bar{y}^2 - \bar{y}^2) \quad (A3.3)$$

Substituting Equation A3.3 into Equation A3.2 we obtain,

$$P(y_0|y_1, y_2, \dots, y_N) = C'' \frac{1}{[\sigma(y_0)]^N} \exp\left\{-\frac{N}{2[\sigma(y_0)]^2}(y_0 - \bar{y})^2\right\} P(y_0) \quad (A3.4)$$

Using the model of $P(y_0)$ given in Equation 5, the final result is obtained:

$$P(y_0|y_1, y_2, \dots, y_N) = C'' \times \frac{1-\epsilon}{[\sigma(0)]^N} \exp\left\{-\frac{N}{2[\sigma(0)]^2}\bar{y}^2\right\} \delta(y_0) + \frac{\epsilon}{y_{0max}[\epsilon(y_0)]^N} \exp\left\{-\frac{N}{2[\sigma(y_0)]^2}(\bar{y}_0 - y)^2\right\}$$

where C'' is given by

$$\frac{1}{C''} = \frac{1-\epsilon}{[\sigma(0)]^N} \exp\left\{-\frac{N}{2[\sigma(0)]^2}\bar{y}^2\right\} + \frac{\epsilon}{y_{0max}} \int_0^{y_{0max}} \frac{dy_0}{[\sigma(y_0)]^N} \exp\left\{-\frac{N}{2[\sigma(y_0)]^2}(y_0 - \bar{y})^2\right\} \quad A3.5$$

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