# Exact Confidence Intervals for Paired Counting Utilizing Modified Bessel Functions of Integral Order

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## Abstract

The original code to determine Neyman-Pearson confidence intervals for extreme low-level, paired counting at the 0.95 confidence level has been extended up to 100 background counts and 100 observed net counts. The code automatically determines confidence intervals of the form [0, xx.xx] for a specified confidence level in the interval [0.5, 0.999]. The exact probability distribution for the net count is expressed in terms of modified Bessel functions of integral order. The code is written in C++ and runs on inexpensive C++ software. Examples of the output of the code are presented. Both the way to extend the code to confidence intervals of the form [yy.yy, xx.xx] and the way to extend the code to background counts and observed net counts of 1000 counts are pointed out.

## INTRODUCTION

Fong and Alvarez (1997) discussed the lack of precision at MDA. Confidence intervals provide a way of expressing a measurement process that lacks precision. Little (1982) discussed the fact that a sample can have a negative net count but must have nonnegative activity.

If x and z are Poisson distributed random variables with expectations a and b, respectively, then the probability of observing k net counts can be expressed as follows (Feller 1966; Haight 1967):

$$P(x-z=k) = \exp(-a-b)\ (\mathrm{sqrt}(a/b))^k\ I_{|k|}(2\mathrm{sqrt}(ab)).$$

In the above equation $I_k$ is the modified Bessel function of the real variable (2sqrt(ab)) and of integral order k. The function $I_k$ is defined by a power series (Feller 1966) and $|k|$ denotes the absolute value of k. If the observed net count and the background count are not greater than 100, the function $I_k$ can be adequately computed using 250 terms in the power series. This fact can be exhibited by looking at the residuals in the well-known recursion relation for the modified Bessel function of integral order (Abramowitz and Stegun 1972).

The original code was developed for extreme low-level, paired counting. Confidence intervals of the form [0, xx.xx] were determined for the expected value of the net count at the 0.95 confidence level (Potter 1999a). Potter (1999b) discussed additional details concerning the application of the original code.

The code discussed in this paper is an extension of the original code; it is valid for the observed background, B (real number), equal to or less than 100.0 and the observed net count, OC (integer), equal to or less than 100. The code discussed in this paper is published

(Potter 2001) and there are no known errors in the code as published. With some insight the code is capable of providing a wide range of confidence intervals for paired counting situations. Additionally, when the background count is taken close to zero the code yields confidence intervals for the expected value of the Poisson distribution that are in agreement with published, tabulated values (Pearson and Hartley 1966).

## NEYMAN-PEARSON CONFIDENCE INTERVALS

Neyman-Pearson confidence intervals (Pearson and Hartley 1966) of the form [0, xx.xx] are automatically determined for the expected value of the net count, for confidence levels gam in the interval [0.50, 0.999]. A further constraint is that B is equal to or greater than 0.00011 when OC = 100 counts and less when OC is less than 100 counts. This constraint is computer system dependent and is required so that exponent overflow does not occur; that is double precision arithmetic allows floating point numbers up to 1.7E308. A statement can be included in the code to terminate the running of the code if B is too small for the value of OC to be used in the computation.

Starting from an overestimate for the desired expected value, this starting value is decreased until that value, with two decimal places, which has probability of observing OC or fewer net counts closest in absolute value to (1-gam) is obtained. The search is made up of a sequence of negative increments of 10.0's, positive increments of 1.0's, negative increments of tenths, and positive increments of hundredths. It should be noted that the probability of observing OC or fewer net counts is a monotone decreasing function of the expected value; <u>there is only one value</u>. Otherwise the search could be much more difficult.

The requirement that gam is equal to or less than 0.999 comes about from the necessity to know the maximum value for the expected value of the net count in order to begin the computation. If gam is made sufficiently large, there will exist a confidence interval of the form [0, xx.xx], with xx.xx non zero. Clearly as gam approaches 1.0, xx.xx increases monotonically. If there does not exist a nonzero value for the expected value, then xx.xx is taken to be zero for the specified gam.

Approximate solutions for Neyman-Pearson confidence intervals follow by approximating the true probability distributions for the net count with Gaussian distributions. Utilizing half-integer corrections and the quadratic formula, approximate confidence intervals can be determined. These intervals give good results when xx.xx and yy.yy are greater than about 10.0; however the time to calculate answers is more than the time to use a modern computer to determine exact solutions.

## COMPUTATIONS

The code was constructed using a personal computer[*†] with Microsoft's VisualC++6.0 (Standard Edition)[‡]. A version of the code has been constructed to run with Borland's

[*] Pentium II-450 microprocessor, Intel Corporation, 3065 Bowers Ave., Santa Clara, CA 95051.
[†] Windows 98, Microsoft Corporation, One Microsoft Way, Redmond, WA 98052-6399.
[‡] Microsoft Corporation, One Microsoft Way, Redmond, WA 98052-6399.

C++Builder 5 (Standard Edition). Borland's C++Builder 5 (Standard Edition)[§] works well and has an advantage of supporting long double computations which is needed to extend the region of validity of this code. The code uses the functions BESI, pow, and nrerror from the original paper. BESI computes values for the modified Bessel functions, pow computes nonnegative integral powers of real numbers, and nrerror handles errors.

To improve the accuracy of the code and to extend the code for large values of OC and small values of B and remain within the limits of double precision arithmetic, the expression $\mathrm{sqrt}((a/b)^k)$ in the original code has been replaced by $(\mathrm{sqrt}(a/b))^k$ throughout this code. Here b is the background, a is the background plus the expected value of the net count, and k ranges from –125 to OC.

The other change to the original code is the number of terms in the left tail; it was increased from including terms as small as the probability of observing -50 net counts to including terms as small as the probability of observing -125 net counts. This increase is caused because the larger values of B allow for smaller values that OC can attain with significant probability. The choice of summing from –125 was based on sensitivity studies.

## EXAMPLES

Upon inputting values of 100.0 for the background, 100 for the observed net count, and 0.999 for the confidence level the code yields the confidence interval [0, 158.4]. As a check on the validity of the computation the probability of observing OC of less counts is also output. In this example the probability of observing 100 or less net counts is equal to 0.000999901502000218. Because the expected value is determined with only two decimal places this probability is usually not identical to (1.0 – gam).

Upon inputting values of 2.0 for the background, -3 for the observed net count and 0.950 for the confidence level the code yields the confidence interval [0, 1.12]. The probability of observing OC or fewer counts is computed to be 0.0499042110624785.

When an 85% confidence interval is sought for the previous example the following output is obtained: "zero; try confidence level greater than 0.899878301587198." At the 85% confidence level the confidence interval is [0, 0] and a confidence level is given above which a nonzero confidence interval may be obtained.

When B becomes close to zero the code yields confidence intervals of the form [0, xx.xx] for the Poisson distribution where OC net counts are observed. In particular when the background is taken to be 0.00011, the observed net count is taken to be 50 and the confidence level is taken to be 0.975, the code yields a confidence interval [0, 65.92]. Finally the probability of OC or less net counts is computed to be 0.0249917061080569.

## CONCLUDING REMARKS

[§] Borland International, Inc., PO Box 660001, Scotts Valley, CA 95067-0001.

The code does a very good job of summing the left tail. Use can be made of this capability to determine the N-P confidence interval of the form [yy.yy, +infinity] for a specified confidence level gam. The real number yy.yy is that value for the expected value that yields a probability of (1.0-gam) of being equal to greater than OC. Equivalently yy.yy is that value of the expected value that has probability gam of being equal to or less than (OC-1). Then two runs of the code enable N-P confidence intervals of the form [yy.yy, xx.xx] to be determined. Most common tables of confidence intervals for the Poisson distribution agree about xx.xx; however some tables use OC and not (OC-1) to determine yy.yy

The lower limit for double precision numbers is 1.7E-308. When B is less than about 0.002, depending on the computer system, exponent underflow may occur. Some computer systems set numbers divided by numbers greater than about 0.588E308 equal to zero; correct results are obtained and no error results. If the system does not handle underflow correctly, it is possible to make the computer program limit the number of terms in the left tail as the background approaches zero; again, the correct results are obtained.

Newer computers with newer software are preferred; overflow and underflow are detected and it is unnecessary to reboot the computer when either overflow or underflow occurs. Newer computers may not run older software; overflow and underflow may lockup the computer. Older computers with older software may give good results. The code of this paper works on a personal computer[**††] and either Borland's Turbo C++ 3.0 for Dos or Turbo C++ for Windows 4.5[‡‡]. Execution is slow.

It is expected that with a newer computer, new software, good familiarity with C++, and a perfect scanner a person could have this program up and running in a few hours.

To extend the code to 1000 background counts and 1000 observed net counts it is necessary to utilize long double precision arithmetic. Also 1800 terms should be used in the function BESI to compute the modified Bessel functions. This condition is determined by using the standard recursion relationship for the modified Bessel functions. Finally the left tail should include terms as small as the probability of observing –500 observed net counts; again, this condition was determined by sensitivity studies.

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