Is the Average of Detected Counts a Good Measure of the Dispersion of Radioactivity Measurements?

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The fluctuations in the decay of long-lived radionuclide are described by the Poisson process (Bateman, 1910). It follows from the Poisson process that the variance of the radioactivity measurement is equal to the average detected counts, $\sigma_x^2 = \mu$, or simply x in the case of a single measurement. This fact makes the statistics of ionizing radiation straightforward to use. Rather than talking about the variance, it is more convenient to define the index of dispersion $\delta_x = \sigma_x^2/\mu$ (known also as the Lexian or Fano factor) as well as the index of variation $v_x = \sigma_x/\mu$, which is a measure of relative error. Thus for a Poisson process one has

$$\delta_x = 1 \; , \tag{1a}$$

$$v_x = 1/\sqrt{\mu} \; . \tag{1b}$$

In the present work we discuss the instances, both known and new, where the Poisson statistics is no longer valid, resulting in the deviations from such a simple picture.

The well known case is when the radionuclide decays substantially during the measurement. Then the underlying statistics is that of the Bernoulli (binomial) process (Von Schweidler, 1905). In this case, however, one has to explicitly incorporate the detection efficiency ε (Ruark and Devol, 1936). The index of dispersion of the measurement for rapidly decaying radionuclide is thus

$$\delta_x = 1 - d\varepsilon \; , \tag{2}$$

where $d = 1 - \exp(-\lambda t)$ is the probability of decay during counting time t and λ is the decay constant. It follows from Eq. 2 that the dispersion, and whence the error of a measurement, is zero, if the radionuclide completely decays during the measurement with a unit efficiency (*i.e.*, $d\varepsilon = 1$).

Another instance of underdispersion ($\delta_x < 1$) is due to the dead-time losses which modify the Poisson statistics. The indices of dispersion for extended (Omote, 1990) and nonextended (Cox, 1962) dead-times are given by Eqs. 3a and 3b, respectively,

$$\delta_x = \begin{cases} 1 - \mu\frac{\tau}{t}\left(2 - \frac{\tau}{t}\right) , & \text{(3a)} \\ \left(1 - \mu\frac{\tau}{t}\right)^2 , & \text{(3b)} \end{cases}$$

where τ is the dead time after each event.

The deviations of δ_x from 1 may also occur when counting the radionuclides decaying in series, such as radon and its daughters (Inkret *et al.* , 1990). The δ_x can be less than 1 or even over 2 depending on a particular equilibrium condition between the member radionuclides, counting time, as well as the detection efficiencies. The equations for δ_x are complicated, however.

In the following we disregard the effects due to dead-time or series decay. Equations 1 and 2 were derived with the assumption that ε and d are constant parameters. However, since the statistics of radioactive decay cannot be decoupled from the statistics of measurement, ε and d have to be considered as variables fluctuating during the measurement. The outcome depends on the rate of fluctuations. Let us consider the efficiency ε first. The efficiency is not constant from count to count because in different events we have different energy losses in an uneven source, different radiation paths and losses on the way to the detector including the detector window, as well as fluctuating processes in the detector itself. Thus, the efficiency is approximately constant in a single event, but its fluctuations are sufficiently rapid so that many changes of ε occur during the measurement time t. Although it sounds counterintuitive, these rapid fluctuations of efficiency have no effect on the statistics of radioactive decay and, as shown by Breitenberger (1955), the average value $\bar{\varepsilon}$ is sufficient as a constant parameter to describe the decay/detection system fluctuating according to either the binomial or Poisson distributions.

The fluctuations with slower rates comparable to the measurement time is another matter. The most obvious examples are variations in sample positioning during multiple measurements and the use of duplicate samples. Other examples are long-term changes in temperature and magnetic field affecting the PMT, long-term variations in gas and air pressure, cosmic-ray bursts in the background *etc.* The fluctuations of the measurement time also belong to the category of slow fluctuations which are present in short measurements with imprecise timing devices, such as multiplexers and multiscalers. Time fluctuations lead to the fluctuations of the probability of decay d (λ is taken as a constant, however (Jakobovits *et al.* , 1995)).

The slow fluctuations lead to overdispersion in the counting data relative to either binomial or Poisson distributions. The overdispersion in the counting data was treated as a symptom by Currie (1972) and recently by Tries (1997). To the best of our knowledge, no formal treatment has been done. Therefore, in this work we consider the effects of slow fluctuations from a point of view of statistical distribution theory. We treat the ε and d as continuous stochastic variables. Using the methods from mathematical statistics (Johnson *et al.* , 1993), we derive the formula for the index of dispersion of counts from rapidly decaying source (modified binomial process)

$$\delta_x = 1 - \bar{d}\bar{\varepsilon} + \mu\left(1 - \frac{1}{N}\right)\left\{v_\varepsilon^2 + (1 + v_\varepsilon^2)\, v_t^2 \left[\frac{(1-\bar{d})\ln(1-\bar{d})}{\bar{d}}\right]^2\right\} , \tag{4}$$

where N is a number of radioactive atoms, $\bar{d} = 1 - \exp(-\lambda\bar{t})$, $\bar{t}$ is the average counting time, and v_ε and v_t are the indices of variation of the efficiency and time fluctuations, respectively. Equation 4 reduces to Eq. 2 when the above fluctuations are zero. Equation 4 predicts that even if $\bar{d}\bar{\varepsilon} = 1$, the error of measurement is not zero, unless $N = 1$.

The modified Poisson limit for slowly decaying source can be obtained from Eq. 4 by setting $N \gg 1$ and $\bar{d} \ll 1$.

$$\delta_x = 1 + \mu\left[v_\varepsilon^2 + (1 + v_\varepsilon^2)\, v_t^2\right] , \tag{5a}$$

$$v_x = \sqrt{1/\mu + v_\varepsilon^2 + (1 + v_\varepsilon^2)\, v_t^2} . \tag{5b}$$

Equations 5 are the generalizations of Eqs. 1 and reduce to them when the efficiency and time fluctuations can be neglected. Equation 1a asserts that the dispersion of radioactivity measurement is unity. However, Eq. (5a) shows that it can be much larger than unity owing to the efficiency and time fluctuations, particularly when the number of detected counts is large. Equation (5b) gives the relative error of radioactivity measurement. Based on the first term under the square root only, the more counts detected, the smaller is the relative error. However, as this term diminishes for a large number of counts, the relative error is essentially determined by the fluctuations of efficiency and time and not by the statistics of radioactive decay. It is not useful to continue the measurement beyond this point.

In summary, we have generalized the well known textbook formulas 1 and 2 (Friedlander *et al.* , 1981) with the formulas 4 and 5, which include the fluctuations in efficiency and counting time. Out of the two, the efficiency fluctuations are more common. So the answer to the question posed in the title can be positive only for long-lived radionuclide when the detection system is extremely stable with respect to efficiency and time, the former being difficult to achieve in a practical counting system.

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